Problem Set 1: Crypto Derivative Pricing

PhD Seminar – Digital Finance Research

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Spring 2026

Due date: Two weeks from distribution.
Total points: 100.
Submission: PDF via course platform. Include all derivations and code.
References: [2], [1], [4], [3].

Problem 1: Characteristic Function of the Kou Model [35 points]

The Kou double-exponential jump-diffusion model specifies the risk-neutral dynamics of a cryptocurrency price S_t as

$$\frac{\mathrm{d}S_t}{S_{t^-}} = (r - q - \lambda\,\zeta)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \,\mathrm{d}\left(\sum_{i=1}^{N_t}(V_i - 1)\right), \tag{1}$$

where:

- r is the risk-free rate, q the continuous dividend yield,
- $\sigma > 0$ is the diffusion volatility,
- N_t is a Poisson process with intensity $\lambda > 0$,
- $V_i > 0$ are i.i.d. jump multipliers with $\ln V_i \sim f_J(y)$,
- $\zeta = \mathbb{E}[V_i - 1]$ is the compensator.

The jump-size density is the double-exponential (asymmetric Laplace):

$$f_J(y) = p\,\eta_1\,e^{-\eta_1 y}\,\mathbb{1}_{y\geq 0} + (1-p)\,\eta_2\,e^{\eta_2 y}\,\mathbb{1}_{y<0}, \tag{2}$$

where $p \in (0,1)$ is the probability of an upward jump, $\eta_1 > 1$ (to ensure $\mathbb{E}[V_i] < \infty$), and $\eta_2 > 0$.

(a) **(20 pts)** Let $X_t = \ln(S_t/S_0)$. Derive the characteristic function

$$\phi_{X_T}(u) \;=\; \mathbb{E}\big[e^{iu\,X_T}\big]$$

in closed form. Proceed as follows:

(i) Write X_T in terms of drift, Brownian, and compound-Poisson components.

(ii) Compute the moment generating function of a single jump $\mathbb{E}[e^{s\,\ln V}]$ and identify the domain of s.

(iii) Use independence of the Brownian and Poisson parts to factor the characteristic function.

(iv) State the final formula explicitly in terms of $(\sigma, \lambda, p, \eta_1, \eta_2, r, q, T)$.

(b) **(15 pts)** Show that the compensator is

$$\zeta = \frac{p\,\eta_1}{\eta_1 - 1} + \frac{(1-p)\,\eta_2}{\eta_2 + 1} - 1$$

and verify that $\phi_{X_T}(u)$ satisfies the martingale condition $\phi_{X_T}(-i) = e^{(r-q)T}$.

Solution Sketch

(a) Under the Kou model, $X_T = \ln(S_T/S_0)$ satisfies

$$X_T = \left(r - q - \tfrac{1}{2}\sigma^2 - \lambda\zeta\right) T + \sigma W_T + \sum_{i=1}^{N_T} J_i,$$

where $J_i = \ln V_i$. The moment generating function of a single jump is

$$M_J(s) = \mathbb{E}[e^{sJ}] = \frac{p\,\eta_1}{\eta_1 - s} + \frac{(1-p)\,\eta_2}{\eta_2 + s}, \quad s \in (-\eta_2, \eta_1).$$

Since W_T and N_T are independent, the characteristic function factors:

$$\phi_{X_T}(u) = \exp\Big(iu\big(r - q - \tfrac{1}{2}\sigma^2 - \lambda\zeta\big)T - \tfrac{1}{2}\sigma^2 u^2 T\Big) \cdot \exp\Big(\lambda T\big(M_J(iu) - 1\big)\Big), \qquad (3)$$

where

$$M_J(iu) = \frac{p\,\eta_1}{\eta_1 - iu} + \frac{(1-p)\,\eta_2}{\eta_2 + iu}.$$

(b) Setting $s = 1$: $\zeta = \mathbb{E}[V - 1] = M_J(1) - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1$. For the martingale check, evaluate $\phi_{X_T}(-i)$, i.e., set $u = -i$ (equivalently $s = 1$), and verify that all jump terms cancel with the compensator, yielding $e^{(r-q)T}$.

Problem 2: Calibration to BTC Options Data [35 points]

You are given a CSV file `btc_options_deribit.csv` containing BTC option data from the Deribit exchange with the following columns:

Column	Description
`strike`	Strike price in USD
`maturity`	Time-to-expiry in years
`type`	`call` or `put`
`mid_iv`	Mid implied volatility (Black–Scholes)
`spot`	BTC spot price at observation
`rf_rate`	Risk-free rate

(a) **(15 pts)** *Kou calibration.* Calibrate the Kou model parameters $\boldsymbol{\theta} = (\sigma, \lambda, p, \eta_1, \eta_2)$ by minimizing the sum of squared implied-volatility errors:

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} \Big(\sigma_{\text{BS}}^{\text{mkt}}(K_i, T_i) - \sigma_{\text{BS}}^{\text{Kou}}(K_i, T_i; \boldsymbol{\theta})\Big)^2,$$

where $\sigma_{\text{BS}}^{\text{Kou}}$ is obtained by:

(i) Pricing the option via the Carr–Madan FFT formula using the characteristic function from Problem 1.

(ii) Inverting the Black–Scholes formula to recover the model-implied volatility.

Report calibrated parameters and RMSE. Plot the implied-volatility smile (market vs. model) for a representative maturity.

(b) **(10 pts)** *Heston comparison.* Repeat the calibration for the Heston stochastic-volatility model with parameters $(\kappa, \bar{v}, \sigma_v, \rho, v_0)$:

$$\mathrm{d}S_t = (r-q)\, S_t \,\mathrm{d}t + \sqrt{v_t}\, S_t \,\mathrm{d}W_t^S, \tag{4}$$

$$\mathrm{d}v_t = \kappa(\bar{v} - v_t)\,\mathrm{d}t + \sigma_v \sqrt{v_t}\,\mathrm{d}W_t^v, \quad \langle \mathrm{d}W^S, \mathrm{d}W^v \rangle = \rho \,\mathrm{d}t. \tag{5}$$

Compare the Kou and Heston fits: which captures the BTC smile better at short vs. long maturities? Present results in a table.

(c) **(10 pts)** *Computational aspects.* Discuss:

(i) Choice of FFT grid parameters (N, α, grid spacing Δ) and their impact on accuracy.

(ii) Local minima issues in calibration. What initial guesses and constraints do you impose? Did you use global optimization (e.g., differential evolution)?

(iii) Sensitivity of calibrated parameters to the data window (use 2–3 different dates if available).

Implementation hint: Use Python with `numpy`, `scipy.optimize`, and `scipy.fft`. A Jupyter notebook template is provided in the course repository.

Solution Sketch

(a) The Carr–Madan call price is

$$C(K,T) = \frac{e^{-\alpha \ln K}}{\pi} \int_0^\infty \Re\Big[e^{-iv \ln K}\, \psi_T(v)\Big]\, \mathrm{d}v,$$

where $\psi_T(v) = \frac{e^{-rT}\,\phi_{X_T}(v-(1+\alpha)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}$ and $\alpha \approx 1.5$ is a dampening parameter. Discretize via FFT with $N = 2^{12}$, $\Delta = 0.25/N$. Use `scipy.optimize.differential_evolution` with bounds $\sigma \in [0.1, 2.0]$, $\lambda \in [0.1, 20]$, $p \in [0.1, 0.9]$, $\eta_1 \in [1.1, 50]$, $\eta_2 \in [0.1, 50]$, then polish with L-BFGS-B.

(b) The Heston characteristic function is

$$\phi^H_{X_T}(u) = \exp\!\big(C(u,T) + D(u,T)\, v_0 + iu(r-q)T\big),$$

with the well-known C and D Riccati solutions (use the "little Heston trap" formulation of Albrecher et al. for numerical stability). Typically Kou captures short-maturity skew better (jumps), while Heston captures term-structure (mean-reverting vol).

(c) Key points: α must satisfy $\alpha > 0$ and $\phi_{X_T}(-(1+\alpha)i) < \infty$. Grid: $N = 4096$, $\eta \cdot \Delta = 2\pi/N$. Global optimization is essential due to non-convexity; differential evolution with `popsize` $= 30$ works well. Parameter stability across dates is a known issue for jump models.

Problem 3: Hedging Effectiveness Analysis [30 points]

Consider a European call option on BTC with strike K, maturity T, priced under the Kou model with your calibrated parameters from Problem 2.

(a) **(10 pts)** *Delta computation.* Derive the delta of the call option under the Kou model. Show that

$$\Delta_{\mathrm{Kou}} = \frac{\partial C}{\partial S} = \frac{e^{-\alpha \ln K}}{\pi S} \int_0^\infty \Re\Big[e^{-iv\ln K}\,(iv+\alpha)\,\psi_T(v)\Big]\,dv.$$

Explain why this differs from the Black–Scholes delta and when the difference is most pronounced.

(b) **(10 pts)** *Simulation study.* Perform a Monte Carlo delta-hedging simulation:

(i) Simulate 10,000 paths of S_t under the Kou model (true DGP) over 30 days, with daily rebalancing ($\Delta t = 1/365$).

(ii) Strategy A: hedge using Δ_{BS} (Black–Scholes delta with flat volatility = calibrated ATM IV).

(iii) Strategy B: hedge using Δ_{Kou}.

(iv) For each path, compute the hedging P&L at expiry: $\mathrm{P\&L} = (S_T - K)^+ - C_0 - \sum_j \Delta_j (S_{t_{j+1}} - S_{t_j})$ (suitably discounted).

Report: mean, standard deviation, 5th and 95th percentiles of P&L for both strategies. Plot the P&L distributions.

(c) **(10 pts)** *Discussion.*

(i) Decompose the residual hedging error into contributions from (i) discrete rebalancing, (ii) jump risk, and (iii) volatility mis-specification.

(ii) How would you reduce the hedging error in practice? Discuss gamma hedging, jump hedging with a second option, and increasing rebalancing frequency (with transaction costs).

(iii) Relate your findings to [4]: which hedging strategies do they recommend for crypto options, and why?

Solution Sketch

(a) Differentiating the Carr–Madan integral w.r.t. S and using $\partial(\ln K)/\partial S = 0$ plus the chain rule on the $e^{-\alpha \ln K}$ term gives the stated formula. The Kou delta differs from BS delta most for short-dated, OTM options where jump risk dominates.

(b) Simulation algorithm:

1. For each time step Δt: $X_{t+\Delta t} = X_t + \mu\Delta t + \sigma\sqrt{\Delta t}\,Z + \sum_{k=1}^{N_{\Delta t}} J_k$, where $Z \sim \mathcal{N}(0,1)$, $N_{\Delta t} \sim \mathrm{Poisson}(\lambda\Delta t)$, $J_k \sim f_J$.
2. Typical results: Strategy B (Kou delta) has ~30–40% lower P&L standard deviation than Strategy A (BS delta).
3. The improvement is largest on jump days and for short maturities.

(c) Key decomposition: The total hedging error is $\epsilon = \epsilon_{\mathrm{discrete}} + \epsilon_{\mathrm{jump}} + \epsilon_{\mathrm{vol}}$. Under daily rebalancing, jump risk (ϵ_{jump}) dominates for crypto. Matić et al. (2023) recommend minimum-variance delta hedging and using variance swaps or a second option to hedge jump/vol risk.

References

[1] Steven L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[2] Steven G. Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8):1086–1101, 2002.

[3] Zhe Li, Wei Zhang, Yi Zhang, and Haoran Yi. Equilibrium pricing of Bitcoin options with stochastic volatility, jumps, and liquidity risk. *Journal of Futures Markets*, 2026.

[4] Jovanka Lili Matić, Natalie Packham, and Lutz Schloegl. Hedging cryptocurrency options. *Review of Derivatives Research*, 26:91–133, 2023.