CBDCs: Mathematical Models and Welfare Analysis

L03 Extended: Formalizing the Economics of Public Digital Money

From deposit market competition to international game theory

Economics of Digital Finance

BSc Course

Welcome Back: From Concepts to Math

[XKCD #2030: "Blockchain"]

Source: xkcd.com/2030 by Randall Munroe, CC BY-NC 2.5

"Today we formalize the economic models behind the concepts introduced in the basic lecture."

Comic sets the stage for our mathematical deep dive into CBDC economics.

From Concepts to Models

What We Know

1. CBDC is a direct liability of the central bank
2. Design trade-offs exist (privacy vs compliance, access vs control)
3. Bank disintermediation (loss of deposits to CBDC) is a key risk
4. CBDC can improve monetary policy transmission (how central bank rate changes flow through to the real economy)

What We'll Formalize

1. Andolfatto (2021): bank competition model with CBDC entry
2. Brunnermeier–Niepelt (2019): equivalence theorem for public vs private money
3. Barrdear–Kumhof (2022): optimal CBDC interest rate
4. Benigno et al. (2022): international CBDC game theory

This lecture builds mathematical foundations for the concepts introduced in the basic CBDC lecture

Mathematical Toolkit

Key Mathematical Concepts

- Utility maximization (choosing the best outcome given constraints)
- First-order conditions (FOC = setting the derivative equal to zero to find the optimum)
- Nash equilibrium (a situation where no player benefits from changing strategy unilaterally)

Notation Preview

Symbol	Meaning
a	Inelastic deposit base (EUR trillions)
b	Deposit rate sensitivity
r_D	Bank deposit interest rate
r_L	Bank lending rate
r_{CBDC}	CBDC interest rate
D	Bank deposits (EUR trillions)
W	Aggregate welfare
V	Country payoff in game

All derivations use BSc-level calculus. Full notation table in Appendix A1

The Bank's Problem – Setup

Model Setup

- Monopoly bank chooses deposit rate r_D to maximize profit
- Deposit supply: $D(r_D) = a + b \cdot r_D$ (depositors supply more when rate is higher)
- Parameters: $a = 5.4$ (EUR trillions, the inelastic deposit base—deposits that stay regardless of rate), $b = 200$ (EUR trillions per unit rate, deposit sensitivity to rate changes)

Worked example: if $r_D = 0.4\%$ then
$D = 5.4 + 200 \times 0.004 = 6.2$ trillion EUR

Bank Profit Function

- $\Pi_B = (r_L - r_D) \cdot D(r_D) - FC$
- Spread $= r_L - r_D$ (the bank's margin on each euro of deposits)
- Lending rate $r_L = 3.5\%$, fixed costs $FC = 0$ (simplified)

Worked example:

$\Pi_B = (0.035{-}0.004){\times}6{,}200\text{B} = 0.031{\times}6{,}200\text{B} = 192.2\text{B EU}$

Based on Andolfatto (2021), combining Klein–Monti monopoly bank with Diamond (1965) government debt

The Bank's Optimal Choice (Pre-CBDC)

Profit Maximization

- FOC: $\dfrac{d\Pi_B}{dr_D} = 0$
- With linear deposit supply:

$$r_D^* = \frac{r_L - a/b}{2}$$

- **Worked example:** $a/b = 5.4/200 = 0.027$, so

$$r_D^* = \frac{0.035 - 0.027}{2} = 0.004 = 0.4\%$$

- At this rate: $D^* = 6.2\text{T EUR}$, $\Pi_B^* = 192.2\text{B EUR}$

Monopoly Distortion

- Monopoly bank pays LESS than competitive rate
- Competitive rate would approach $r_L = 3.5\%$ (zero profit)
- Gap $= r_L - r_D^* = 3.5\% - 0.4\% = 3.1$ percentage points measures monopoly power
- Depositors lose surplus; bank captures it as profit

The monopoly bank restricts deposit rates below the competitive level, just as a monopolist restricts output

Enter CBDC – The Outside Option

CBDC as Competitive Constraint

- Central bank introduces CBDC paying $r_{CBDC} = 1.0\%$
- Depositors now have an outside option: if $r_D < r_{CBDC}$, they switch
- CBDC acts as a floor on deposit rates

Bank's New Problem

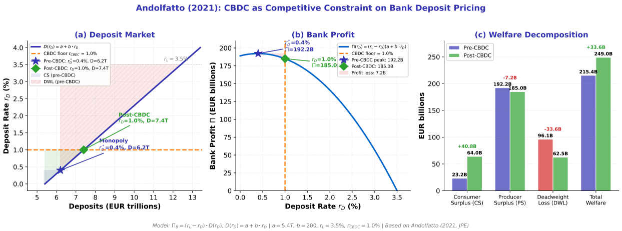
- Bank must set $r_D \geq r_{CBDC}$ or lose all deposits
- Monopoly optimum was $r_D^* = 0.4\%$, but $r_{CBDC} = 1.0\% > 0.4\%$, so CBDC floor **binds**
- Bank forced to raise rate to $r_D = 1.0\%$

Worked example:

- $D(0.01) = 5.4 + 200 \times 0.01 = 7.4$T EUR (up from 6.2T)
- $\Pi_{post} = (0.035 - 0.01) \times 7{,}400\text{B} = 185.0\text{B}$ EUR (down from 192.2B)
- Higher $r_D \Rightarrow$ higher $D \Rightarrow$ more lending $\Rightarrow$ more inclusion

CBDC disciplines the bank like a new competitor entering the market—depositors benefit from the threat

Andolfatto Equilibrium Comparison



- Panel (a): CBDC floor at 1.0% forces the bank above its monopoly optimum of 0.4%, shifting equilibrium from 6.2T to 7.4T deposits
- Panel (b): Bank profit falls from 192.2B to 185.0B EUR as the monopoly margin is compressed, but total deposits increase by 1.2 trillion
- Panel (c): Consumer surplus rises significantly; total welfare increases because the deposit market moves closer to the competitive outcome

Andolfatto (2021) shows CBDC may increase total deposits despite reducing bank profit

Andolfatto Key Results

Surprising Results

- CBDC does NOT necessarily reduce lending
- Higher deposit rates ⇒ more deposits ⇒ more funds to lend
- Bank profit falls but depositor welfare rises
- Net welfare effect depends on parameters

Numerical Summary

Variable	Pre-CBDC	Post-CBDC
Deposit rate	0.4%	1.0%
Deposits	6.2T EUR	7.4T EUR
Bank profit	192.2B	185.0B
Lending	6.2T	7.4T
Profit change	–	−7.2B (−3.7%)

Key insight: CBDC can be pro-competitive, increasing deposits and lending while reducing monopoly rents

The Equivalence Question

The Core Question

- If people swap bank deposits for CBDC, does it matter for the economy?
- Modigliani–Miller (a theorem showing that, under ideal conditions, how a firm is financed does not affect its value) analogy: does the *composition* of money matter?
- Brunnermeier & Niepelt (2019) answer: under specific conditions, NO

Intuition

- Deposits fund bank lending. If deposits leave for CBDC, banks lose funding
- BUT: central bank can lend those CBDC funds back to banks (pass-through funding)
- Net effect: banks' funding source changes, but total credit unchanged
- Like refinancing a mortgage—same house, different lender

Brunnermeier & Niepelt (2019), Journal of Monetary Economics. An equivalence result for private vs public money

The Equivalence Theorem

Theorem (Simplified)

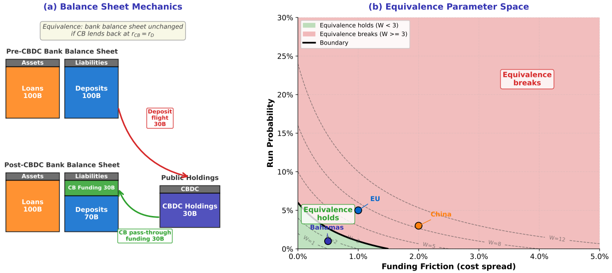
- If: (1) central bank provides pass-through funding at same terms, (2) no bank runs, (3) no friction differences
- Then: swapping deposits for CBDC leaves prices, output, and allocation unchanged
- Formally: equilibrium under $(D, 0_{CBDC}) =$ equilibrium under $(D - x, x_{CBDC})$ for any x

Worked Example

- Economy: 100B deposits, 0 CBDC
- Swap: 30B deposits move to CBDC
- Central bank lends 30B back to banks at same rate
- Banks: 70B deposits + 30B CB funding = 100B total funding
- Lending unchanged: still 100B
- Outcome identical

The theorem shows CBDC need not cause a credit crunch—IF the central bank provides pass-through funding

When Equivalence Breaks Down



Brunnermeier & Niepelt (2019, JME 106): Eq(D,0) = Eq(D−x, x) requires $r_{CB} = r_D$, no bank runs, no friction differences.

- Panel (a): Balance sheet mechanics showing how pass-through funding preserves total bank funding despite deposit-to-CBDC migration
- Panel (b): Parameter space showing where equivalence holds (green) vs breaks down (red)—frictions, bank runs, and funding term mismatches break the result
- Real-world implication: most economies have frictions that partially break equivalence, making CBDC design choices consequential

Equivalence breaks when: banks face funding cost premium, depositors have heterogeneous preferences, or bank runs are possible

Policy Implications of (Non-)Equivalence

If Equivalence Holds

- CBDC is neutral for financial system
- Design choices are about convenience, not stability
- Central bank can freely issue CBDC without worrying about credit
- Holding limits unnecessary

If Equivalence Breaks (Likely)

- CBDC design matters for credit supply
- Holding limits protect against disintermediation
- Pass-through funding design is critical
- Rate setting affects bank profitability

Worked example: if bank funding premium $= 0.5\%$ after CBDC, lending rate rises 0.5%, credit contracts by $\sim 10\%$

Most CBDC researchers believe equivalence partially breaks—hence the careful design with holding limits and no interest

The Optimization Problem

Central Bank's Objective

- Maximize welfare:

$$W = \text{GDP gain} + \text{Inclusion gain} - \text{Disintermediation cost}$$

- GDP gain from Barrdear & Kumhof (2022): $\Delta Y = f(\theta)$ where $\theta =$ CBDC share of money supply
- Calibration: $\theta = 30\% \Rightarrow \Delta Y = +3\%$ GDP permanently

The Trade-off

- Higher $r_{CBDC} \Rightarrow$ more CBDC adoption (θ rises)
- More adoption $\Rightarrow$ GDP benefits from lower transaction costs
- But also $\Rightarrow$ deposit flight, bank funding stress, potential credit contraction
- Optimal r_{CBDC} balances these forces

Worked example: if r_{CBDC} too high (3.5%), disintermediation cost dominates ($W = -3.451$). If $r_{CBDC} = 0\%$, $W = 0$ (no benefit). Sweet spot near 0.8%

Based on Barrdear & Kumhof (2022), JEDC. DSGE model calibrated to pre-2008 US economy

Deriving the Optimal Rate

Simplified Welfare Function

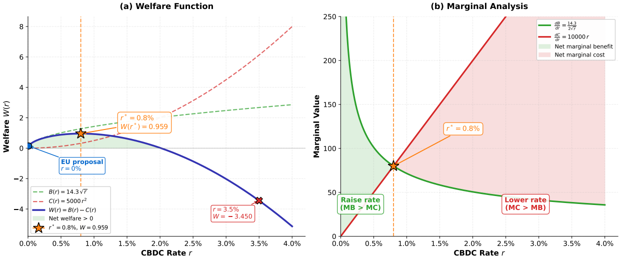
- $W(r) = B(r) - C(r)$
- $B(r) = k_1\sqrt{r}$ (concave: diminishing returns to higher CBDC rate)
- $C(r) = k_2 r^2$ (convex: accelerating disintermediation costs)
- FOC: $\frac{dW}{dr} = \frac{dB}{dr} - \frac{dC}{dr} = 0$
- $\frac{dB}{dr} = \frac{k_1}{2\sqrt{r}}; \quad \frac{dC}{dr} = 2k_2 r$

Worked Example

- Parameters: $k_1 = 14.3$, $k_2 = 5{,}000$
- Setting FOC: $\frac{14.3}{2\sqrt{r}} = 2 \times 5{,}000 \times r$
- $14.3 = 20{,}000 \cdot r^{3/2}$
- $r^{3/2} = 0.000715$
- $r^* = 0.000715^{2/3} = 0.008 = 0.8\%$
- Check: $W(0.8\%) = 14.3\sqrt{0.008} - 5{,}000 \times 0.008^2$
- $= 1.279 - 0.320 = 0.959$
- Compare: $W(0\%) = 0$, $W(3.5\%) = 2.674 - 6.125 = -3.451$

The optimal CBDC rate is typically low—explaining why most central banks propose 0% initially

Visualizing the Optimum



- The welfare curve shows an inverted-U shape: welfare rises with low CBDC rates, peaks at $r^* = 0.8\%$, then turns sharply negative as disintermediation costs dominate
- The optimal rate r^* sits where marginal benefit equals marginal cost—the classic optimization condition
- Sensitivity: the optimal rate shifts left (lower) when the banking sector is fragile, and right (higher) when financial inclusion is poor

Key policy result: the optimal CBDC rate is positive but modest, typically 0–1% depending on economy characteristics

Sensitivity Analysis

Factors Pushing r^* Higher

- High unbanked rate (strong inclusion gains)
- Inefficient banking sector (high monopoly rents)
- Strong digital infrastructure (low CBDC deployment costs)
- Example: Nigeria (62% unbanked) $\Rightarrow r^*$ higher than EU (2% unbanked)

Factors Pushing r^* Lower

- Fragile banking sector (high disintermediation risk)
- High bank concentration (severe credit contraction)
- Low digital literacy (CBDC adoption slow anyway)
- Example: EU (concentrated banking) $\Rightarrow r^*$ lower, hence 0% initial rate

The Digital Euro at 0% is consistent with optimal policy for an economy with low unbanked rate and fragile bank margins

Deposit Market Welfare – Setup

Consumer Surplus (CS)

- CS = depositor welfare = benefit depositors get above the rate they receive
- Monopoly bank pays $r_D^* <$ competitive rate $\Rightarrow$ low CS
- $r_{max} = r_L$ in the competitive limit (at r_L the bank earns zero spread)

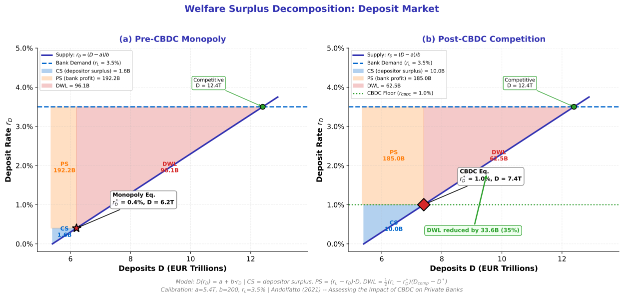
Producer Surplus (PS) and Deadweight Loss (DWL)

- $PS = (r_L - r_D) \cdot D(r_D)$ (bank profit, with $FC = 0$)
- $DWL = \frac{1}{2}(r_D^{comp} - r_D^{mono})(D^{comp} - D^{mono})$
- Monopoly $\Rightarrow$ high PS, low CS, positive DWL

Worked example (pre-CBDC):
$DWL = \frac{1}{2} \times (0.035 - 0.004) \times (12.4\text{T} - 6.2\text{T}) = 96.1\text{B}$
Post-CBDC:
$DWL = \frac{1}{2} \times (0.035 - 0.01) \times (12.4\text{T} - 7.4\text{T}) = 62.5\text{B}$
DWL reduction: $96.1 - 62.5 = 33.6$B welfare gain

Welfare analysis uses standard consumer/producer surplus framework applied to the deposit market

Welfare Decomposition Chart



- Panel (a): Pre-CBDC monopoly—large producer surplus, small consumer surplus, significant deadweight loss triangle ($DWL = 96.1\text{B}$)
- Panel (b): Post-CBDC—CBDC floor at 1% compresses monopoly margin, expanding consumer surplus and reducing deadweight loss to 62.5B
- Net welfare change: DWL falls by 33.6B—the pie gets bigger, not just redistributed

Deadweight loss reduction is the key welfare argument for CBDC—it corrects monopoly distortion in the deposit market

Who Wins, Who Loses?

Winners

Group	Gain	Mechanism
Depositors	+33.6B	DWL recaptured, higher rates
Unbanked	New access	Digital payment inclusion
Treasury	Seigniorage	Revenue on CBDC issuance

Losers

Group	Loss	Mechanism
Bank shareholders	−7.2B	Margin compression
Privacy advocates	Hard to quantify	Surveillance risk

Key point: net welfare positive because DWL reduction (33.6B) far exceeds profit loss (7.2B)

Like any policy reform, CBDC creates winners and losers. The welfare case depends on total gains exceeding total losses

Welfare Under Uncertainty

The Tail Risk

- Normal times: CBDC improves welfare (deposit market competition)
- Crisis times: CBDC may amplify instability (instant flight to safety)
- $E[W] = P(\text{normal}) \cdot W_{\text{normal}} + P(\text{crisis}) \cdot W_{\text{crisis}}$

Numerical Example

- $P(\text{normal}) = 0.95$, $W_{\text{normal}} = +33.6\text{B}$
- $P(\text{crisis}) = 0.05$, $W_{\text{crisis}} = -400\text{B}$ (severe bank run loss)
- $E[W] = 0.95 \times 33.6 + 0.05 \times (-400) = 31.92 - 20.0 = +11.92\text{B}$

With holding limits: $W_{\text{crisis}} = -100\text{B}$ (capped flight)

- $E[W] = 0.95 \times 33.6 + 0.05 \times (-100) = 31.92 - 5.0 = +26.92\text{B}$
- Holding limits more than double expected welfare!

Holding limits are welfare-optimal because they cap the downside risk of digital bank runs while preserving normal-time benefits

[XKCD #927: "Standards"]

Source: xkcd.com/927 by Randall Munroe, CC BY-NC 2.5

"International CBDC competition risks the same outcome—more currencies, not more interoperability. Let's model why."

The game theory of CBDC competition explains why coordination is hard but necessary.

Currency Competition as a Game

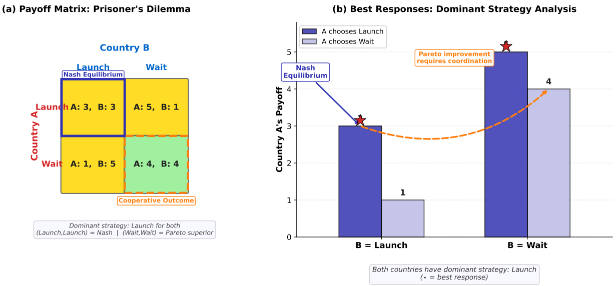
Setup

- Two countries (A and B) each decide: Launch CBDC or Wait
- Payoffs depend on both countries' decisions
- First-mover advantage ($\gamma =$ bonus payoff for launching while rival waits): early launcher may capture cross-border payments
- Network effects (the phenomenon where a product becomes more valuable as more people use it): more users $\Rightarrow$ more valuable
- Based on Benigno, Schilling & Uhlig (2022) framework

Payoff Drivers

- Benefits of launching: seigniorage, sanctions power, policy autonomy, payment efficiency
- Costs of launching: development cost, bank disruption, privacy backlash
- Strategic interaction: if rival launches first, your currency faces substitution pressure

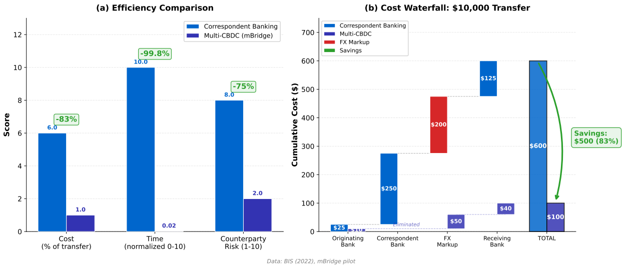
International CBDC competition is a strategic game—each country's optimal action depends on what others do



- Panel (a): The payoff matrix reveals a Prisoner's Dilemma structure—both countries launching is NOT the best joint outcome, but each has an incentive to launch regardless
- Panel (b): Reaction functions show the Nash equilibrium (named after John Nash—the outcome where no country can improve its payoff by unilaterally changing strategy)
- The Nash equilibrium (both launch) may be welfare-inferior to coordination (both wait), explaining calls for international CBDC standards

Nash equilibrium predicts mutual CBDC launch; international coordination could achieve the Pareto-superior outcome

Cross-Border Settlement



- Panel (a): Multi-CBDC settlement dramatically reduces cost (−83%), time (−99.8%), and counterparty risk (−75%) compared to correspondent banking (the traditional system where banks maintain accounts at each other to process cross-border payments)
- Panel (b): The cost waterfall shows where savings come from—eliminating intermediary bank fees and FX markup generates the largest gains
- Benigno, Schilling & Uhlig (2022): cross-border CBDC use enforces interest rate synchronization, limiting monetary policy autonomy—an extension of the impossible trinity (a country cannot simultaneously maintain a fixed exchange rate, free capital movement, and independent monetary policy)

Source: BIS (2022) data. Wholesale CBDC reduces cross-border payment costs from 6% to under 1%

Model Synthesis

Converging Insights

- Andolfatto: CBDC is pro-competitive (0.4% $\rightarrow$ 1.0%, +1.2T deposits)
- Brunnermeier–Niepelt: no credit crunch if pass-through
- Barrdear–Kumhof: optimal rate is low but positive ($r^* = 0.8\%$)
- Game theory: international coordination needed (Prisoner's Dilemma)

Policy Design Principles

Principle	Model Basis
Start with 0% rate	Barrdear–Kumhof: uncertainty favors caution
Holding limits essential	Welfare: doubles expected welfare
Pass-through funding	Brunnermeier–Niepelt conditions
International coordination	Benigno et al.: Prisoner's Dilemma
Two-tier distribution	Andolfatto: preserves bank role

All four models converge on a common recommendation: cautious CBDC introduction with strong safeguards

Open Questions

Unresolved Issues

- Optimal holding limit level (3,000 EUR? 5,000? Dynamic?)
- Long-run bank adaptation (do banks find new business models?)
- Privacy-efficiency trade-off (can zero-knowledge proofs resolve it?)
- Digital divide implications (does CBDC worsen inequality?)

Your Assignment Connection

- These models are the foundation for Exercise 6 (Disintermediation Calculator)
- Exercise 7 (Tiered Remuneration) applies the optimal rate framework
- Quiz questions 9–14 test understanding of these models
- Research paper topic: any of the open questions

CBDCs remain an active research frontier. The models here provide the analytical toolkit for evaluating new proposals

References

- Andolfatto, D. (2021). Assessing the Impact of Central Bank Digital Currency on Private Banks. *The Economic Journal*, 131(634), 525–540.
- Barrdear, J. & Kumhof, M. (2022). The macroeconomics of central bank digital currencies. *Journal of Economic Dynamics and Control*, 142, 104148.
- Benigno, P., Schilling, L.M. & Uhlig, H. (2022). Cryptocurrencies, currency competition, and the impossible trinity. *Journal of International Economics*, 136, 103601.
- Brunnermeier, M.K. & Niepelt, D. (2019). On the equivalence of private and public money. *Journal of Monetary Economics*, 106, 27–41.
- BIS (2022). Project mBridge: Connecting economies through CBDC. *BIS Innovation Hub*.
- Bindseil, U. (2020). Tiered CBDC and the financial system. *ECB Working Paper No. 2351*.

Full references for all models and data sources cited in this lecture

Appendix A1: Complete Notation Table

Symbol	Meaning	Section
D	Bank deposits (EUR trillions)	Andolfatto
r_D	Bank deposit interest rate	Andolfatto
r_{CBDC}	CBDC interest rate	Andolfatto / Optimal Rate
r_L	Bank lending rate	Andolfatto
r_{max}	Maximum deposit rate (competitive limit)	Welfare
a	Inelastic deposit base (EUR trillions)	Andolfatto
b	Deposit rate sensitivity (EUR trillions per unit rate)	Andolfatto
Π_B	Bank profit (EUR billions)	Andolfatto
FC	Fixed costs	Andolfatto
$U(c)$	Utility of consumption	Welfare
α	Risk aversion parameter	Welfare
W	Aggregate welfare	Welfare / Optimal Rate
$B(r)$	Benefit function of CBDC rate	Optimal Rate
$C(r)$	Cost function of CBDC rate	Optimal Rate
k_1	Benefit scaling parameter ($= 14.3$)	Optimal Rate
k_2	Cost scaling parameter ($= 5{,}000$)	Optimal Rate
CS	Consumer surplus (depositor welfare)	Welfare
PS	Producer surplus (bank profit)	Welfare
DWL	Deadweight loss	Welfare
$L(D)$	Lending as function of deposits	Equivalence
σ	Volatility / uncertainty parameter	Welfare / Game Theory
Y	GDP / output	Optimal Rate
θ	CBDC share of money supply	Optimal Rate
γ	First-mover advantage payoff bonus	Game Theory
V_i	Country i payoff in CBDC game	Game Theory
δ	Discount factor (present value weight)	Game Theory

Reference page for all mathematical notation used in this lecture

Appendix A2: Andolfatto Full Derivation

Pre-CBDC FOC

$$\Pi_B = (r_L - r_D)(a + b \cdot r_D)$$
$$= r_L \cdot a + r_L \cdot b \cdot r_D - a \cdot r_D - b \cdot r_D^2$$

$$\frac{d\Pi_B}{dr_D} = r_L \cdot b - a - 2b \cdot r_D = 0$$

$$r_D^* = \frac{r_L \cdot b - a}{2b} = \frac{r_L - a/b}{2}$$

With $a = 5.4,\ b = 200,\ r_L = 0.035$:

$$r_D^* = \frac{0.035 - 0.027}{2} = 0.004$$

Post-CBDC (Kuhn–Tucker)

- $\max \Pi_B \quad \text{s.t. } r_D \geq r_{CBDC}$
- If $r_D^* < r_{CBDC}$: constraint binds, $r_D = r_{CBDC}$
- If $r_D^* \geq r_{CBDC}$: constraint slack, monopoly solution unchanged
- With $r_{CBDC} = 0.01 > r_D^* = 0.004$: binds.

$$r_D = 0.01$$

Post-CBDC equilibrium:

$$D = 5.4 + 200 \times 0.01 = 7.4\text{T}$$
$$\Pi_B = (0.035 - 0.01) \times 7{,}400\text{B} = 185.0\text{B}$$

Complete derivation for students who want the mathematical details

Appendix A3: Brunnermeier–Niepelt Sufficient Conditions

Three Sufficient Conditions

1. **Pass-through funding:** CB lends to banks at rate $r_{CB} = r_D$ (no funding cost premium)
2. **No bank runs:** depositors do not panic-withdraw (Diamond–Dybvig stability)
3. **No friction asymmetry:** CBDC and deposits face identical regulatory, tax, and convenience costs

Proof Sketch

- Under conditions 1–3, bank balance sheets adjust: deposits $\downarrow$, CB funding $\uparrow$, total funding unchanged
- Loan supply unchanged $\Rightarrow$ real allocation unchanged
- Modigliani–Miller logic: liability composition is irrelevant when frictions are absent
- Policy implication: making conditions hold (e.g., credible pass-through) is the design goal

Formal conditions from Brunnermeier & Niepelt (2019), Journal of Monetary Economics, 106, 27–41