Monetary Economics of Digital Currencies: Mathematical Models and Empirical Analysis

L02 Extended: Formalizing Money Theory for the Crypto Age

From Baumol-Tobin cash management to Diamond-Dybvig bank runs

Economics of Digital Finance

BSc Course

Welcome Back: From Concepts to Math

"In God we trust; all others must bring data."

— W. Edwards Deming

In L02 Basic, you learned **what** money does and **why** Bitcoin struggles as money.

Now we formalize **how much** cash to hold, **when** bank runs happen, and **how volatile** crypto really is — with math that makes predictions.

This extended lecture adds four formal models to the conceptual framework from L02 Basic

From Concepts to Models

L02 Basic: Concepts You Know

1. **MV=PY**: Money supply times velocity equals price level times real output — the equation of exchange
2. **Volatility problem**: Bitcoin's price swings too much for everyday pricing
3. **Gresham's Law**: "Bad money drives out good" — people spend depreciating currency and hoard appreciating currency
4. **Stablecoins**: Designed to maintain a fixed price (peg) relative to a reference currency like USD

L02 Extended: Models We Add

1. **Baumol-Tobin**: How much cash should you hold? Optimal balance between transaction costs and interest foregone
2. **Diamond-Dybvig**: When do stablecoins collapse? A coordination game (where outcomes depend on what everyone else does) with two equilibria
3. **GARCH**: How do we measure and forecast crypto volatility? Time-varying variance that clusters
4. **Currency substitution**: Why do people switch from local currency to crypto? A utility-based (satisfaction-maximizing) model

Each model builds on one concept from L02 Basic; together they form a complete analytical toolkit

Mathematical Toolkit

Variables We Use

Symbol	Meaning
M	Money supply (total money in circulation)
V	Velocity (how fast money changes hands)
P	Price level (average price of goods)
Y	Real output (total goods and services)
i	Interest rate (cost of holding cash)
c, b	Transaction cost per withdrawal
W	Wealth (total assets owned)
r	Return (profit rate on an investment)
T	Time horizon (number of periods)
n	Number of withdrawals

Math Operations You Need

Square root: $\sqrt{x}$ finds the number that, multiplied by itself, gives x. Example: $\sqrt{9} = 3$.

Optimization: Finding the value of a variable that makes a function as large or small as possible. We write "minimize TC(n)" meaning "find the n that makes total cost smallest."

Elasticity: How sensitive one variable is to changes in another, measured in percentages. If interest rates rise 10% and cash holdings fall 5%, the elasticity is -0.5.

Equilibrium: A state where no one wants to change their behavior. A ball at the bottom of a bowl is in stable equilibrium; a ball on top of a hill is in unstable equilibrium.

We introduce formal notation (including Greek letters) starting in Section 2; this slide establishes the building blocks

The Cash Management Problem

The Setup (Baumol, 1952; Tobin, 1956)
You earn Y dollars per year. You can keep money in:

- A **checking account** (earns 0% interest, but you can spend instantly)
- A **savings account** (earns interest rate i, but costs b dollars each time you transfer money out)

The trade-off:

- Hold more cash $\rightarrow$ lose interest income
- Hold less cash $\rightarrow$ pay more withdrawal fees

New notation: We now introduce σ (sigma, the Greek letter) to denote volatility (the standard deviation of returns, measuring how much a price swings around its average).

Why This Matters for Crypto

- Transaction cost b varies enormously:
  - Traditional bank: $b \approx \$2$ (ATM fee, time)
  - Digital bank: $b \approx \$0.50$ (app transfer)
  - Crypto wallet: $b \approx \$0.10$ (gas fee on L2)
- Lower b means you hold **less** cash (more frequent, cheaper transfers)
- Crypto's low b should increase velocity V — consistent with the velocity puzzle from L02 Basic

Key question: How much cash should you hold at any moment?

Baumol-Tobin is the first formal model of money demand; it predicts how technology changes cash-holding behavior

Deriving Optimal Cash Holdings

Total Cost Function
If you make n withdrawals per year, each of size Y/n:

$$TC(n) = \underbrace{b \cdot n}_{\text{withdrawal fees}} + \underbrace{\frac{i \cdot Y}{2n}}_{\text{interest foregone}}$$

(First term: you pay b dollars each of n times. Second term: average cash balance is Y/(2n), and you lose i percent interest on it.)

Minimize by setting $\frac{dTC}{dn} = 0$:

$$b - \frac{iY}{2n^2} = 0 \quad \Rightarrow \quad n^* = \sqrt{\frac{iY}{2b}}$$

Optimal cash holdings:

$$\boxed{M^* = \frac{Y}{2n^*} = \sqrt{\frac{bY}{2i}}}$$

Worked Example
Given: $Y = \$36{,}000/\text{year}$, $i = 5\%$, $b = \$2$

Step 1: Optimal withdrawals

$$n^* = \sqrt{\frac{0.05 \times 36{,}000}{2 \times 2}} = \sqrt{450} = 21.2$$

Step 2: Optimal cash holdings

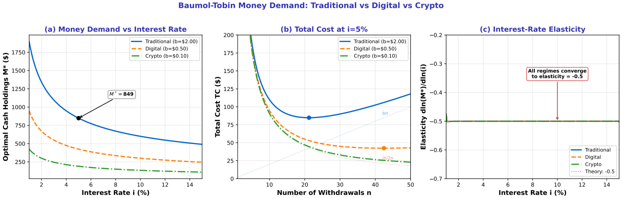
$$M^* = \sqrt{\frac{2 \times 36{,}000}{2 \times 0.05}} = \sqrt{720{,}000} = \$848.53$$

Step 3: Total cost at optimum

$$TC^* = \sqrt{2bYi} = \sqrt{2 \times 2 \times 36{,}000 \times 0.05} = \$84.85$$

So you should hold about \$849 in cash at any time, making about 21 withdrawals per year.

The square-root formula shows money demand rises with income and transaction costs, but falls with interest rates



- **Panel (a):** Lower transaction costs (crypto b=\$0.10) dramatically reduce optimal cash holdings compared to traditional banking (b=\$2).
- **Panel (b):** At i=5%, the crypto user's total cost curve is almost flat — many small withdrawals cost almost nothing.
- **Panel (c):** All three regimes converge to elasticity -0.5: a 10% rate increase always reduces cash demand by 5%.

Crypto's low transaction costs predict higher velocity and lower cash balances — exactly what we observe empirically

Key Results and Crypto Implications

Elasticities (How Sensitive?)

Interest-rate elasticity of money demand:

$$\frac{\partial \ln M^*}{\partial \ln i} = -0.5$$

(A 10% increase in interest rates reduces optimal cash holdings by 5%, regardless of transaction costs.)

Income elasticity:

$$\frac{\partial \ln M^*}{\partial \ln Y} = +0.5$$

(A 10% increase in income raises optimal cash holdings by 5%.)

Transaction cost elasticity:

$$\frac{\partial \ln M^*}{\partial \ln b} = +0.5$$

(Higher withdrawal costs mean you hold more cash to avoid frequent trips.)

Implications for Digital Money

1. **Low b → high velocity:** Crypto makes transfers nearly free, so people hold less idle cash and transact more frequently
2. **DeFi yield → low M^*:** When DeFi offers high interest (i), rational agents minimize cash holdings
3. **Staking as savings:** Crypto staking (locking tokens to earn rewards) is analogous to the savings account in Baumol-Tobin
4. **Gas fees as b:** Ethereum gas fees function exactly as b — when gas spikes, users batch transactions and hold more ETH

Limitation: Model assumes deterministic income. Crypto income is volatile, requiring extensions (precautionary savings motive).

Baumol-Tobin explains why crypto ecosystems with low fees and high yields exhibit dramatically higher velocity than traditional money

Bank Runs: The Diamond-Dybvig Setup

The Model (Diamond & Dybvig, 1983)
Three time periods: $t = 0, 1, 2$

- t=0: Everyone deposits \$1 in the bank
- t=1: Some depositors need money early ("early consumers")
- t=2: Remaining depositors withdraw with returns

Two types of depositors:

- **Type 1** (fraction λ, Greek letter "lambda"): Need money at t=1. Receive $c_1 = 1$ (their deposit back)
- **Type 2** (fraction $1-\lambda$): Can wait until t=2. Receive $c_2 = R > 1$ (deposit plus profit)

The bank invests in a long-term project returning $R > 1$ at t=2, but only $L < 1$ if liquidated early at t=1.

The Key Insight: Coordination
The bank's problem:

- It promised $c_1 = 1$ to early withdrawers
- It invested in illiquid (hard to sell quickly) assets returning R at t=2
- If too many withdraw at t=1, the bank must liquidate assets at loss ($L < 1$)

This is a coordination game:

- If you believe others will NOT run $\rightarrow$ you do not run $\rightarrow$ bank survives
- If you believe others WILL run $\rightarrow$ you must run too $\rightarrow$ bank fails

Your best action depends entirely on what you believe other depositors will do — not on the bank's actual solvency (whether its assets exceed liabilities).

Diamond-Dybvig shows bank runs are rational: even a perfectly solvent bank can fail if depositors lose confidence simultaneously

The Two Equilibria

Good Equilibrium (No Run)

- Only Type 1 depositors (fraction λ) withdraw at $t{=}1$
- Bank has enough liquid assets to pay them $c_1 = 1$
- Type 2 depositors wait, receive $c_2 = R > 1$
- Everyone is better off than under autarky (self-reliance without a bank)

Payoff: Type 1 gets 1, Type 2 gets R.

Bad Equilibrium (Bank Run)

- ALL depositors (both types) try to withdraw at $t{=}1$
- Bank must liquidate long-term assets at fire-sale price $L < 1$
- Total available: $\lambda \cdot 1 + (1{-}\lambda) \cdot L < 1$ per depositor
- First in line get paid; latecomers get nothing

Payoff: Everyone gets $L < 1$ on average (some get 0).

Why Both Are Equilibria

In each case, no individual wants to deviate given what others do. If no one runs, running costs you the return R. If everyone runs, not running means you get nothing.

The bad equilibrium is self-fulfilling: the run causes the insolvency that depositors feared, even though the bank was solvent before the run

Diamond-Dybvig Applied to Stablecoins

Stablecoins as Banks

The analogy is exact:

- **Depositors** = Stablecoin holders
- **Bank assets** = Reserves (Treasury bonds, cash, commercial paper)
- **Withdrawal** = Redemption (exchanging stablecoin for USD)
- $R > 1$ = Interest earned on reserves
- $L < 1$ = Fire-sale price of illiquid reserves

Terra/UST Collapse (May 2022)

- UST was algorithmic: no real reserves, just LUNA token backing
- When UST depegged slightly, holders rushed to redeem
- Minting LUNA to pay redemptions crashed LUNA's price
- Death spiral: UST depeg → LUNA crash → worse depeg
- \$40B+ destroyed in one week

Phase Dynamics Formalization

We model the withdrawal rate x (fraction of holders trying to redeem) and reserve ratio R:

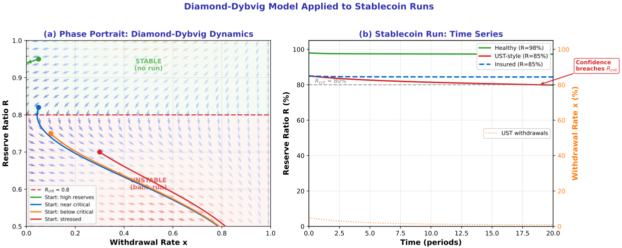
$$\frac{dx}{dt} = -k(R - R_{crit}) \cdot x(1 - x)$$

Where:

- x = fraction of holders withdrawing (0 to 1)
- R = current reserve ratio (reserves/liabilities)
- R_{crit} = critical threshold (typically 80%)
- k = sensitivity to reserve shortfall

Interpretation:

- When $R > R_{crit}$: $dx/dt < 0$ (withdrawals shrink, confidence holds)
- When $R < R_{crit}$: $dx/dt > 0$ (withdrawals accelerate, run begins)
- $x(1-x)$ ensures S-shaped transition (slow start, rapid middle, saturation)



- **Panel (a):** Phase portrait shows two regions. Above R_{crit}=0.80, trajectories flow toward stability (no run). Below R_{crit}, trajectories flow toward full collapse.
- **Panel (b):** Time series comparison. The UST-style scenario (starting at R=0.85) breaches R_{crit} and collapses rapidly. Deposit insurance (dashed) prevents the run.

The critical insight: once reserves breach R_{crit}, collapse is self-reinforcing and nearly impossible to stop without external intervention

Deposit Insurance and Holding Limits

How Insurance Eliminates the Bad Equilibrium
Diamond & Dybvig's key policy result:

- Government deposit insurance guarantees repayment
- If depositors **know** they will be repaid, they have no reason to run
- The bad equilibrium disappears — only the good equilibrium remains
- The insurance may never need to be paid out (the guarantee itself prevents runs)

In the model:

$$\text{With insurance: } R_{eff} = R + \Delta_{insurance}$$

where $\Delta_{insurance}$ is the perceived government backing, ensuring $R_{eff} > R_{crit}$ always.

Can We Insure Stablecoins?

- **Fiat-backed (USDT, USDC):** Possible. Regulate as narrow banks (banks that hold only safe, liquid assets), require full reserves, provide limited insurance
- **Crypto-backed (DAI):** Harder. Over-collateralization (150%+) acts as self-insurance, but crypto collateral itself is volatile
- **Algorithmic (UST):** Impossible. No real assets to insure against. The "insurance" would require unlimited money printing

CBDC Alternative:

- CBDCs (Central Bank Digital Currencies) are inherently insured by the sovereign
- No bank run possible: central bank is the issuer
- But: holding limits may be needed to prevent disintermediation (people moving all deposits from banks to CBDC, starving banks of funding)

Diamond-Dybvig's policy conclusion applies directly: credible guarantees eliminate runs, but algorithmic stablecoins cannot provide them

GARCH(1,1): Modeling Time-Varying Volatility

The Problem

In L02 Basic, we noted Bitcoin's volatility is 50–85% annually. But volatility is not constant — it **clusters**:

- After a big price move (up or down), more big moves follow
- Calm periods tend to persist too
- Standard deviation measured over a fixed window misses this pattern

GARCH(1,1) Model (Bollerslev, 1986)

Conditional variance (today's expected volatility given yesterday's information):

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Where:

- ω (omega) = baseline variance (always positive)
- α (alpha) = reaction to yesterday's shock ϵ_{t-1}^2
- β (beta) = persistence of yesterday's variance
- $\epsilon_t = \sigma_t \cdot z_t$, where z_t is a standard normal random variable

Key Properties

Stationarity condition:

$$\alpha + \beta < 1$$

(If this fails, volatility explodes to infinity over time.)

Unconditional (long-run) variance:

$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

(The average level volatility returns to after shocks die out.)

Persistence: $\alpha + \beta$ measures how slowly volatility returns to its long-run level.

- $\alpha + \beta = 0.95$: shock half-life $\approx$ 14 days
- $\alpha + \beta = 0.99$: shock half-life $\approx$ 69 days

Intuition: GARCH says "if yesterday was crazy, today is probably crazy too" — and quantifies exactly how much.

GARCH is the standard volatility model in finance: Engle won the 2003 Nobel Prize for the original ARCH model (Engle, 1982)

Calibrating GARCH to Bitcoin

Bitcoin GARCH(1,1) Parameters
Estimated from daily BTC/USD returns:

- $\omega = 10^{-5}$ (small baseline)
- $\alpha = 0.10$ (moderate shock reaction)
- $\beta = 0.85$ (high persistence)
- Persistence: $\alpha + \beta = 0.95$

Unconditional daily variance:

$$\bar{\sigma}^2 = \frac{10^{-5}}{1 - 0.95} = 2 \times 10^{-4}$$

Annualized volatility:

$$\bar{\sigma}_{ann} = \sqrt{2 \times 10^{-4}} \times \sqrt{252} = 22.4\%$$

(Multiply daily vol by $\sqrt{252}$ because there are 252 trading days per year.)

S&P 500 for Comparison

- $\omega = 2 \times 10^{-6}$ (smaller baseline)
- $\alpha = 0.08$ (slightly lower reaction)
- $\beta = 0.88$ (slightly higher persistence)
- Persistence: $\alpha + \beta = 0.96$

Unconditional annualized vol:

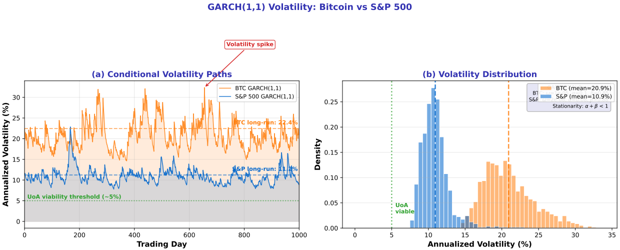
$$\bar{\sigma}_{ann} = \sqrt{\frac{2 \times 10^{-6}}{0.04}} \times \sqrt{252} = 11.2\%$$

Comparison Table

	BTC	S&P
Ann. vol	22.4%	11.2%
Persistence	0.95	0.96
ω	10^{-5}	2×10^{-6}

BTC is $\sim$2x more volatile but has similar persistence.

Bitcoin's annualized volatility of 22.4% is about twice the S&P 500's, but both show strong volatility clustering ($\alpha + \beta > 0.95$)



- **Panel (a):** Simulated conditional volatility paths. BTC (orange) fluctuates more and spikes higher, but both return to their long-run averages.
- **Panel (b):** Volatility distributions. BTC's distribution has a longer right tail (more extreme volatility episodes). The green line marks the ∼5% threshold for unit-of-account viability.

Both assets use GARCH(1,1) with seed=42; BTC's higher ω drives a higher baseline, while similar persistence produces comparable clustering patterns

Volatility and the Unit-of-Account Function

Why Volatility Kills Unit-of-Account
For money to serve as a unit of account (standard pricing benchmark):

- Prices must be **meaningful** over time
- A coffee priced at 0.001 BTC today might cost 0.0005 or 0.002 BTC tomorrow
- "Menu costs" (repricing expenses) become enormous
- Contracts denominated in volatile currency carry unhedgeable risk (risk that cannot be eliminated through financial instruments)

Threshold estimate: Annual volatility must be below ~5% for practical unit-of-account use. This is based on:

- Major fiat currencies: 3–8% annual vol
- Consumer tolerance for price uncertainty

GARCH Forecasting for UoA Assessment
From our GARCH model, we can forecast:

$$\sigma^2_{t+h} = \bar{\sigma}^2 + (\alpha + \beta)^h(\sigma^2_t - \bar{\sigma}^2)$$

(Future volatility decays toward unconditional level, with speed depending on persistence.)

BTC as UoA?

- Unconditional vol: 22.4% $\gg$ 5% threshold
- Even in calm periods: $\sigma_{min} \approx$ 10–15%
- Conclusion: BTC cannot serve as unit of account under current volatility regime

Stablecoins as UoA?

- USDT vol: ~0.5–2% (under normal conditions)
- Passes the 5% threshold easily
- But: tail risk from de-peg events (see Section 3)

GARCH quantifies exactly why Bitcoin fails as a unit of account: its unconditional volatility is 4–5x above the practical threshold

CES Utility Framework

A consumer uses two currencies (A and B) and derives utility (satisfaction) from the liquidity services (ease of transacting) each provides:

$$U = \left[\alpha \cdot m_A^\rho + (1-\alpha) \cdot m_B^\rho\right]^{1/\rho}$$

Where:

- $m_A, m_B =$ real balances of currency A and B
- $\alpha =$ preference weight for currency A
- ρ (rho) $=$ substitution parameter ($-\infty < \rho \leq 1$)
- Elasticity of substitution: $\sigma_s = \frac{1}{1-\rho}$

Key cases:

- $\rho \to 1$: Perfect substitutes ($\sigma_s \to \infty$), meaning any small advantage causes complete switching
- $\rho \to 0$: Cobb-Douglas ($\sigma_s = 1$)
- $\rho \to -\infty$: Perfect complements ($\sigma_s \to 0$)

Application to Crypto Adoption

Setting A = local fiat, B = crypto:

The optimality condition:

$$\frac{m_A}{m_B} = \left(\frac{\alpha}{1-\alpha}\right)^{\sigma_s} \left(\frac{i_B + \pi_B}{i_A + \pi_A}\right)^{\sigma_s}$$

(Ratio of holdings depends on relative costs: interest rate i plus inflation π for each currency.)

Implications:

- High local inflation (π_A) shifts demand toward crypto
- High crypto yield (i_B from staking/DeFi) attracts holdings
- When σ_s is large, small cost differences cause large shifts

Empirical example: Argentina ($\pi_A > 100\%$, 2023) saw rapid crypto adoption because $\pi_A \gg \pi_B$, making crypto the lower-cost money despite volatility.

CES utility = Constant Elasticity of Substitution; it nests perfect substitutes, Cobb-Douglas, and perfect complements as special cases

Gresham's Law: A Logit Formalization

From Aphorism to Model
Gresham's Law ("bad money drives out good") can be modeled as a spending probability:

$$P(\text{spend}_A) = \frac{1}{1 + \exp\left(-\gamma(E[r_B] - E[r_A])\right)}$$

Where:

- $P(\text{spend}_A)$ = probability of spending currency A
- $E[r_A], E[r_B]$ = expected returns on A, B
- γ (gamma) = sensitivity to return differential

Interpretation:

- If $E[r_B] > E[r_A]$: people hoard B ("good money"), spend A ("bad money")
- γ controls how sharply behavior switches (higher γ = more decisive agents)
- At $E[r_B] = E[r_A]$: each currency equally likely to be spent

Crypto Application
Set A = stablecoins, B = Bitcoin:

- $E[r_A] \approx 0\%$ (stablecoins maintain peg)
- $E[r_B] \approx 50\%+$ (Bitcoin expected appreciation)
- $\Rightarrow P(\text{spend}_A) \gg P(\text{spend}_B)$

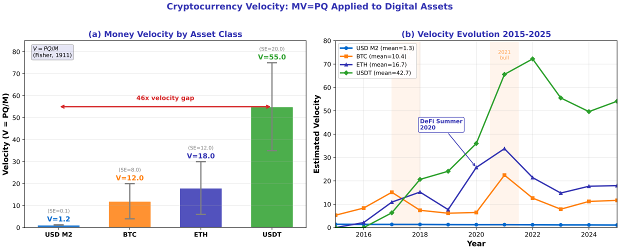
This explains:

1. Why stablecoins dominate crypto payments while Bitcoin is hoarded ("HODL" culture)
2. Why Bitcoin velocity is lower than expected despite high transaction volume
3. Why "crypto-ization" in developing countries uses stablecoins for transactions but Bitcoin for savings

Important caveat: Classical Gresham's Law requires a fixed exchange rate. In crypto, exchange rates float, so the mechanism operates through expected appreciation rather than legal tender laws.

The logit model formalizes Gresham's Law as a continuous probability rather than a binary switch, capturing gradual adoption patterns

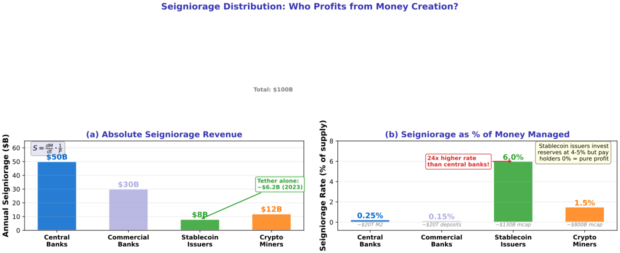
Crypto Velocity: Empirical Estimation



- **Panel (a):** Velocity varies enormously: USDT ($V \approx 55$) circulates 46x faster than USD M2 ($V \approx 1.2$). Error bars reflect high estimation uncertainty for crypto.
- **Panel (b):** Velocity spikes during bull markets (2017, 2021) when speculative trading surges. USD velocity has steadily declined since 2015.

Based on Samani (2017) and Fisher (1911); crypto velocity estimates use on-chain transaction volume divided by circulating supply

Seigniorage Distribution Across Money Types



- **Panel (a):** Absolute seigniorage: Central banks earn the most ($50B), but stablecoin issuers earn $8B from a much smaller base.
- **Panel (b):** As a percentage of money managed, stablecoin issuers earn 6% — **24x** the rate of central banks. They invest reserves at 4–5% interest while paying holders 0%.

Seigniorage $S = (dM/dt) \cdot (1/P)$; stablecoin issuers capture monopoly rents that CBDCs would return to the public

The History of Money Standards

"Money is a matter of functions four:
a medium, a measure, a standard, a store."

— Alfred Milnes, *The Economics of Commerce* (1919)

Gold standard → Bretton Woods → Fiat float → **Crypto?**

Each transition changed who earns seigniorage
and who bears the costs of monetary instability.

The gold standard constrained seigniorage to the cost of mining; fiat freed governments to print; crypto distributes seigniorage to miners and stakers

Applying Portfolio Theory to Money

Markowitz (1952) portfolio selection applies to money choice:

- Each form of money has expected return $E[r]$ and risk σ
- Rational agents choose the portfolio on the efficient frontier (the set of portfolios offering maximum return for each level of risk)
- "Money demand" = position on the frontier matching investor risk tolerance

Asset Parameters:

Asset	$E[r]$	σ
Cash	0%	0%
Savings	3%	1%
CBDC	1%	0.5%
USDT	0%	5%
BTC	50%	65%
ETH	40%	75%

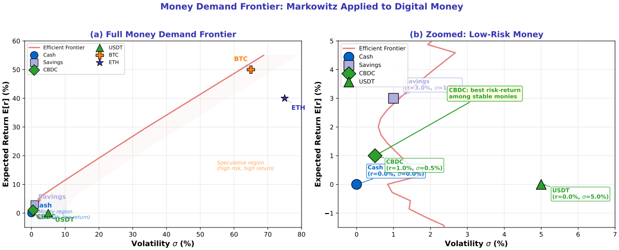
Key Insights

1. **Traditional money clusters near origin:** Cash, savings, and CBDCs all have very low risk and low return. They are "money" precisely because they are boring.
2. **Crypto is in the speculative zone:** BTC and ETH offer high expected returns but with enormous risk. They behave more like venture capital than money.
3. **USDT is inefficient:** It has the risk of crypto (de-peg risk) but the return of cash (0%). It sits below the efficient frontier.
4. **CBDC dominates USDT:** CBDC offers higher return (potential interest), lower risk (sovereign backing), and sits closer to the frontier.

This framework explains why different "monies" coexist: agents with different risk preferences choose different points on the frontier.

Markowitz's insight that risk-return trade-offs determine asset allocation applies equally to money: agents "invest" in the form of money matching their risk tolerance

Money Demand Frontier: Scatter and Frontier



- **Panel (a):** Full frontier showing the dramatic separation between traditional money (near origin) and crypto assets (upper right). The red line is the efficient frontier computed from all assets.
- **Panel (b):** Zoomed view of the low-risk "money region." CBDC offers the best risk-return trade-off among stable monies. USDT sits below the frontier (inefficient).

Frontier computed using Markowitz mean-variance optimization with mild short-selling allowed; CBDC parameters are projections based on proposed designs

Taxonomy by Three Dimensions

1. **Issuer:**
   - Central bank (CBDC, physical cash)
   - Commercial bank (deposits)
   - Private company (stablecoins)
   - Decentralized protocol (BTC, ETH)
2. **Technology:**
   - Account-based (identity verified; bank accounts, some CBDCs)
   - Token-based (possession proves ownership; cash, crypto)
   - Hybrid (CBDC designs combining both)
3. **Backing:**
   - Full reserve (1:1 asset backing)
   - Fractional reserve (partial backing, leverage)
   - Algorithmic (no backing, code-based)
   - None (pure fiat, Bitcoin)

Cross-Reference to Our Models

Model	What It Tells Us
Baumol-Tobin (Sec. 2)	Token-based money with low b increases velocity
Diamond-Dybvig (Sec. 3)	Fractional reserve and algorithmic backing are frag full reserve or sovereign backing eliminates runs
GARCH (Sec. 4)	Decentralized, unbacked money (BTC) has high volatility, failing UoA
CES Substitution (Sec. 5)	When fiat fails (π_A high), agents substitute tow any available alternative

For analysis of systemic contagion across these money types, see L08 Extended (Systemic Risk).

The classification framework combines issuer, technology, and backing dimensions; each of our four models illuminates different combinations

Model Synthesis: Four Lenses on Digital Money

How the Models Connect

1. **Baumol-Tobin** predicts *how much* money agents hold
   - Low crypto $b \to$ low $M^* \to$ high V
2. **Diamond-Dybvig** predicts *when* money systems fail
   - $R < R_{crit} \to$ self-fulfilling run
3. **GARCH** measures *how volatile* money is
   - $\sigma_{ann} > 5\% \to$ fails UoA
4. **CES substitution** predicts *which* money wins
   - High π_A + low $\pi_B \to$ substitution

Together: Low transaction costs (1) increase velocity but also fragility (2). High volatility (3) limits adoption as money but not as speculation. Currency competition (4) disciplines monetary policy.

Policy Synthesis

Policy	Model Support
Regulate stablecoins as banks	Diamond-Dybvig: require full reserves
Issue CBDCs	Markowitz: dominates USDT; D-D: elimi runs
Impose holding limits	D-D: prevent disintermediation
Allow crypto as asset, not money	GARCH: too volatile for UoA/MoE
Tax crypto seigniorage	Seigniorage theory: private capture of publi

For systemic contagion analysis across traditional and crypto financial systems, see L08 Extended.

Each policy recommendation is grounded in a specific model prediction, not mere opinion; this is the power of formal economic analysis

Open Questions in Monetary Economics of Crypto

Unresolved Theoretical Questions

1. **Will Bitcoin volatility decline?** GARCH persistence ($\alpha + \beta = 0.95$) is high but ω could fall as markets mature. If $\omega \to 0$, unconditional vol $\to 0$. But is this realistic?
2. **Can algorithmic stablecoins ever work?** Diamond-Dybvig says no without exogenous (externally provided) insurance. But what about hybrid designs with partial reserves?
3. **Optimal CBDC interest rate?** Too high: disintermediates banks. Too low: cannot compete with stablecoins. What does the Baumol-Tobin framework say about the optimal rate?

Unresolved Empirical Questions

1. **True crypto velocity:** On-chain data overestimates (wash trading) and underestimates (off-chain transactions). How do we measure V correctly?
2. **Seigniorage redistribution:** If stablecoin issuers must share interest with holders, does the business model survive? How elastic is stablecoin demand to yield?
3. **Currency substitution thresholds:** At what inflation rate does crypto adoption become mainstream? Argentina ($> 100\%$) suggests the threshold is high, but Nigeria ($\sim 30\%$) shows earlier adoption.

Research opportunity: Each question is an active area of academic research with publication potential.

These questions define the frontier of monetary economics research; answering them requires combining theory (our four models) with empirical crypto data

References

Core Models

- Baumol, W. (1952). "The Transactions Demand for Cash: An Inventory Theoretic Approach." *Quarterly Journal of Economics*, 66(4), 545–556.
- Tobin, J. (1956). "The Interest-Elasticity of Transactions Demand for Cash." *Review of Economics and Statistics*, 38(3), 241–247.
- Diamond, D. & Dybvig, P. (1983). "Bank Runs, Deposit Insurance, and Liquidity." *Journal of Political Economy*, 91(3), 401–419.
- Bollerslev, T. (1986). "Generalized Autoregressive Conditional Heteroskedasticity." *Journal of Econometrics*, 31(3), 307–327.

Foundations and Applications

- Engle, R. (1982). "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation." *Econometrica*, 50(4), 987–1007.
- Fisher, I. (1911). *The Purchasing Power of Money*. Macmillan.
- Samani, K. (2017). "Understanding Token Velocity." Multicoin Capital Research.
- Yermack, D. (2015). "Is Bitcoin a Real Currency? An Economic Appraisal." In *Handbook of Digital Currency*, 31–43.
- Markowitz, H. (1952). "Portfolio Selection." *Journal of Finance*, 7(1), 77–91.

All readings available on course platform; Diamond & Dybvig (1983) is essential for understanding stablecoin fragility

A1: Complete Notation Reference

Latin Symbols

Symbol	Definition
M	Money supply
M^*	Optimal cash holdings (Baumol-Tobin)
V	Velocity of money (PQ/M)
P	Price level
Y	Real output (or annual income)
i	Nominal interest rate
b	Transaction cost per withdrawal
n	Number of withdrawals
n^*	Optimal number of withdrawals
TC	Total cost function
R	Reserve ratio (Diamond-Dybvig) or gross return
R_{crit}	Critical reserve threshold
S	Seigniorage revenue
W	Wealth
c_1, c_2	Consumption at period 1, 2
L	Liquidation value (< 1)
x	Withdrawal rate (fraction)
k	Sensitivity parameter
r	Return on asset
$E[r]$	Expected return

Greek Symbols

Symbol	Definition
σ	Volatility (standard deviation)
σ^2	Variance
σ_t^2	Conditional variance at time t
$\bar{\sigma}^2$	Unconditional (long-run) variance
ω	GARCH constant term (omega)
α	ARCH coefficient (alpha): shock reaction
β	GARCH coefficient (beta): persistence
ϵ_t	Return innovation (shock) at time t
λ	Fraction of early consumers (D-D)
ρ	CES substitution parameter (rho)
σ_s	Elasticity of substitution
π	Inflation rate
π^e	Expected inflation
γ	Logit sensitivity parameter (gamma)

Greek letters are introduced gradually: Section 2 introduces σ, Section 3 adds λ, Section 4 adds $\omega, \alpha, \beta, \epsilon$

A2: Baumol-Tobin Complete Derivation

Step 1: Setup
Income Y arrives once per year. You withdraw n times, each of size $W = Y/n$.
Average cash balance: $\bar{M} = W/2 = Y/(2n)$

Step 2: Total Cost

$$TC(n) = \underbrace{bn}_{\text{fee cost}} + \underbrace{i \cdot \frac{Y}{2n}}_{\text{interest cost}}$$

Step 3: First-Order Condition

$$\frac{dTC}{dn} = b - \frac{iY}{2n^2} = 0$$

$$n^2 = \frac{iY}{2b} \quad \Rightarrow \quad n^* = \sqrt{\frac{iY}{2b}}$$

Step 4: Optimal Holdings

$$M^* = \frac{Y}{2n^*} = \frac{Y}{2} \cdot \sqrt{\frac{2b}{iY}} = \sqrt{\frac{bY}{2i}}$$

Step 5: Second-Order Condition

$$\frac{d^2 TC}{dn^2} = \frac{iY}{n^3} > 0 \quad \checkmark$$

(Confirms n^* is a minimum, not maximum.)

Step 6: Elasticities

$$\frac{\partial \ln M^*}{\partial \ln i} = \frac{\partial}{\partial \ln i}\left[\frac{1}{2}\ln b + \frac{1}{2}\ln Y - \frac{1}{2}\ln 2 - \frac{1}{2}\ln i\right] = -\frac{1}{2}$$

Similarly: $\partial \ln M^*/\partial \ln Y = +1/2$, $\partial \ln M^*/\partial \ln b = +1/2$.

Step 7: Total Cost at Optimum

$$TC^* = b\sqrt{\frac{iY}{2b}} + \frac{iY}{2}\sqrt{\frac{2b}{iY}} = \sqrt{2bYi}$$

This derivation uses only calculus (derivatives and square roots); no advanced mathematics required

A3: GARCH(1,1) Stationarity and Unconditional Variance

Unconditional Variance Derivation

Start with GARCH(1,1):

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Take unconditional expectation (long-run average):

$$E[\sigma_t^2] = \omega + \alpha E[\epsilon_{t-1}^2] + \beta E[\sigma_{t-1}^2]$$

Since $\epsilon_t = \sigma_t z_t$ and $z_t \sim N(0,1)$:

$$E[\epsilon_t^2] = E[\sigma_t^2 \cdot z_t^2] = E[\sigma_t^2] \cdot E[z_t^2] = E[\sigma_t^2]$$

In stationarity, $E[\sigma_t^2] = E[\sigma_{t-1}^2] = \bar{\sigma}^2$:

$$\bar{\sigma}^2 = \omega + \alpha\bar{\sigma}^2 + \beta\bar{\sigma}^2$$

$$\bar{\sigma}^2(1 - \alpha - \beta) = \omega$$

$$\boxed{\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}}$$

Stationarity Condition

For $\bar{\sigma}^2$ to be positive and finite:

$$1 - \alpha - \beta > 0 \quad \Leftrightarrow \quad \alpha + \beta < 1$$

If $\alpha + \beta \geq 1$: the denominator is zero or negative, and variance is infinite or undefined. The process is "integrated" (IGARCH) — shocks never die out.

Half-Life of Volatility Shocks

After a shock, conditional variance decays as:

$$\sigma_{t+h}^2 - \bar{\sigma}^2 = (\alpha + \beta)^h(\sigma_t^2 - \bar{\sigma}^2)$$

Half-life (time for shock to decay by 50%):

$$h_{1/2} = \frac{\ln(0.5)}{\ln(\alpha + \beta)}$$

Examples:

- BTC ($\alpha+\beta = 0.95$): $h_{1/2} = 13.5$ days
- S&P ($\alpha+\beta = 0.96$): $h_{1/2} = 17.0$ days

Stationarity is crucial: without it, GARCH forecasts are meaningless because volatility has no level to revert to