Expectation and Moments

Lesson 08

Digital Finance

Outline

Expected Value

Expected value (mean) of random variable:

Discrete:

$$E[X] = \sum_x x \cdot p(x)$$

Continuous:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Interpretation:

- Long-run average
- Center of mass of distribution

Expected value is the probability-weighted average.

Properties of Expectation

Linearity:

$$E[aX + b] = aE[X] + b$$

$$E[X + Y] = E[X] + E[Y] \quad \text{(always true!)}$$

For independent X and Y:

$$E[XY] = E[X] \cdot E[Y]$$

Function of RV:

$$E[g(X)] = \sum_x g(x) \cdot p(x) \quad \text{or} \quad \int g(x) \cdot f(x)\, dx$$

Linearity holds even for dependent variables!

Variance

Variance measures spread around the mean:

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Standard deviation:

$$\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$$

Properties:

- $\text{Var}(X) \geq 0$
- $\text{Var}(aX + b) = a^2\text{Var}(X)$
- $\text{Var}(X) = 0$ iff X is constant

Variance = risk in finance.

Covariance

Covariance: Do two variables move together or opposite?

- $\text{Cov}(X, Y) > 0$: when X goes up, Y tends to go up
- $\text{Cov}(X, Y) < 0$: when X goes up, Y tends to go down
- $\text{Cov}(X, Y) = 0$: no linear relationship (but could still be related!)

Why it matters – Variance of sum:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

In words: If X and Y move opposite (negative covariance), combining them reduces total variance – this is diversification!

Formula: Cov$(X, Y) = E[XY] - E[X]E[Y]$. Scale-dependent; see correlation next.

Correlation

Correlation standardizes covariance:

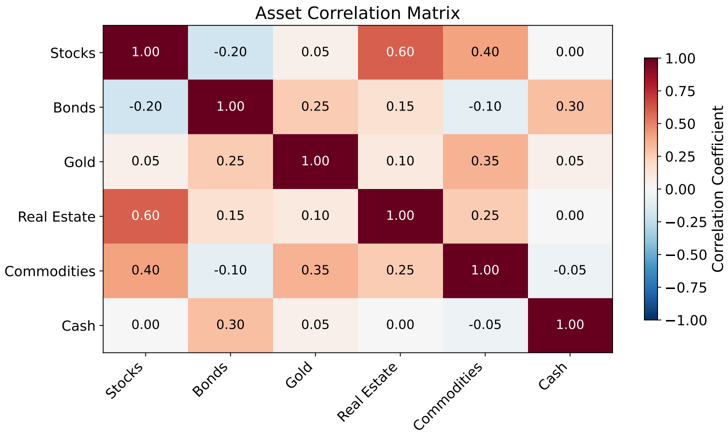
$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Properties:

- $-1 \leq \rho \leq 1$
- $\rho = 1$: perfect positive linear relationship
- $\rho = -1$: perfect negative linear relationship
- $\rho = 0$: no linear relationship

Correlation measures strength and direction of linear relationship.

Correlation Matrix



Diversification works best with low or negative correlations.

Portfolio Return

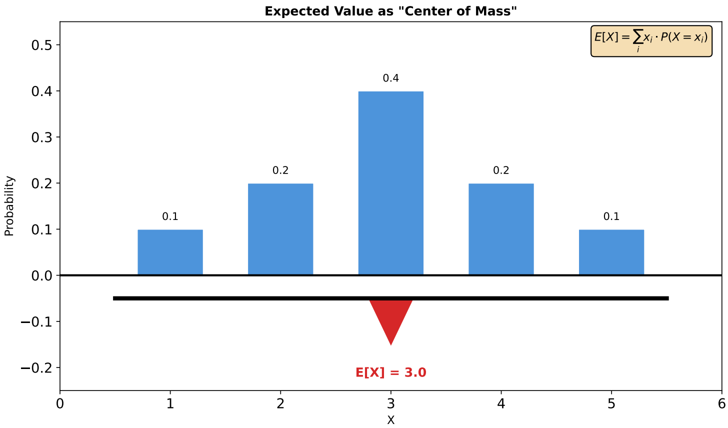
Weights = fraction of money in each asset (must sum to 1).
Simple example: 50% in Stock A, 50% in Stock B

- Portfolio return = 0.5 × (A's return) + 0.5 × (B's return)
- Expected portfolio return = 0.5 × E[A] + 0.5 × E[B]

Portfolio variance (the key insight):
Var(Portfolio) = (weighted variances) + (covariance terms)
Result: If A and B move opposite ($\rho < 0$), portfolio variance can be *less* than either asset alone!

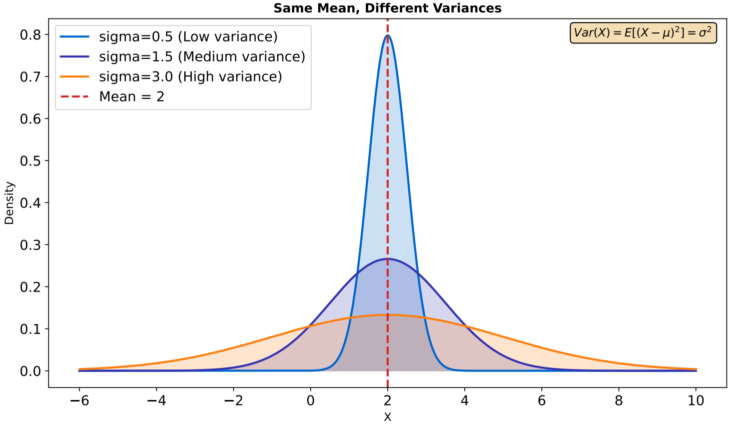
This is why diversification works: negative covariances cancel out risk.

Expected Value: Center of Mass



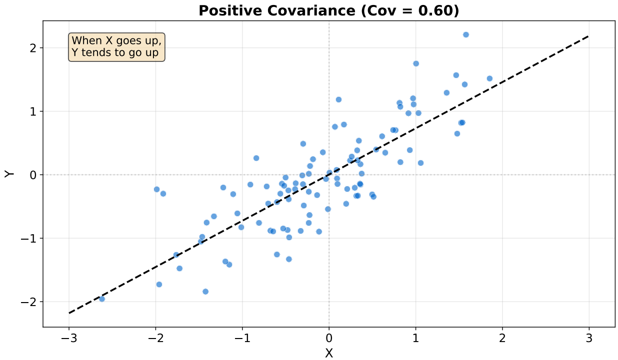
E[X] is the balance point of the distribution.

Variance: Measuring Spread



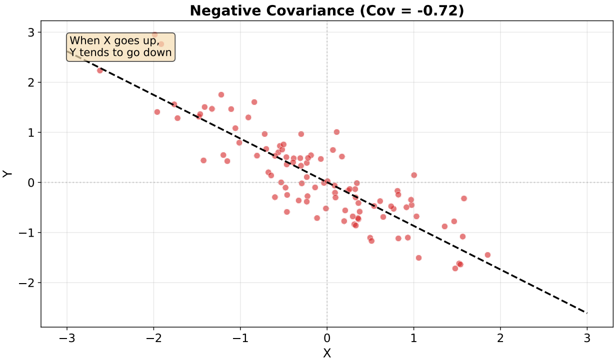
Higher variance means more spread around the mean.

Positive Covariance



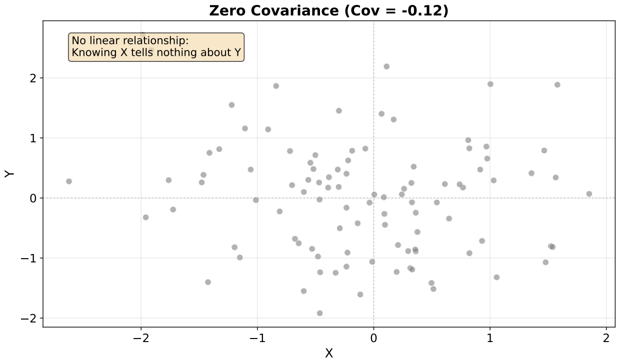
When X goes up, Y tends to go up (positive relationship).

Negative Covariance



When X goes up, Y tends to go down (inverse relationship).

Zero Covariance



No linear relationship: knowing X tells you nothing about Y.

Summary

Expectation:

- Probability-weighted average
- Linear: $E[aX + bY] = aE[X] + bE[Y]$

Variance:

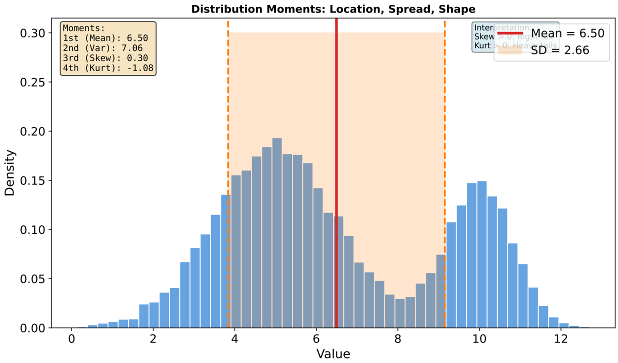
- Measures spread/risk
- $\text{Var}(aX) = a^2\text{Var}(X)$

Covariance/Correlation:

- Measure co-movement
- Critical for portfolio theory

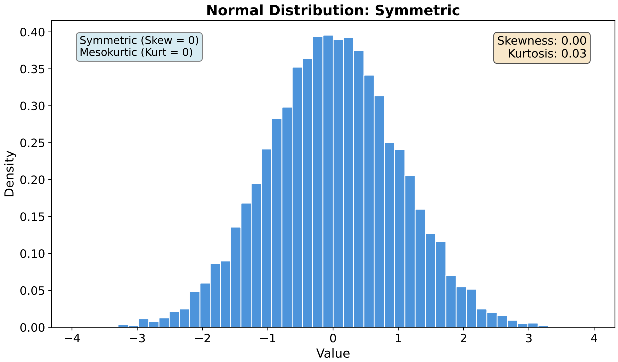
Next lesson: Sampling and Central Limit Theorem

Moment Hierarchy



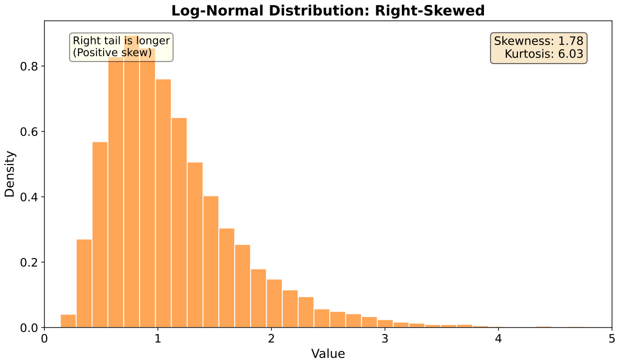
Moments progressively describe location, spread, shape, and tails.

Normal Distribution: Symmetric



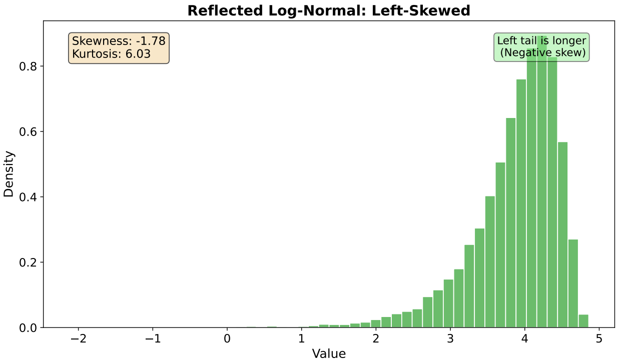
Symmetric distribution has skewness near zero.

Right-Skewed Distribution



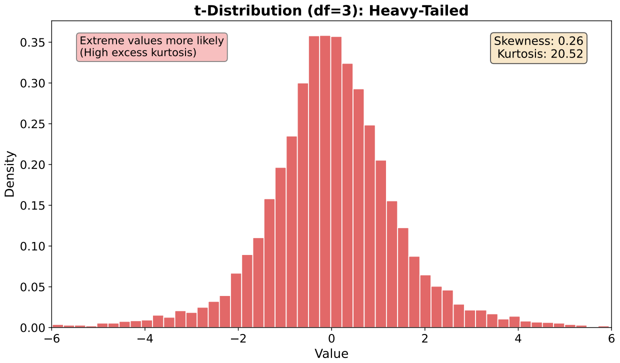
Right (positive) skew: long right tail, mean ¿ median.

Left-Skewed Distribution



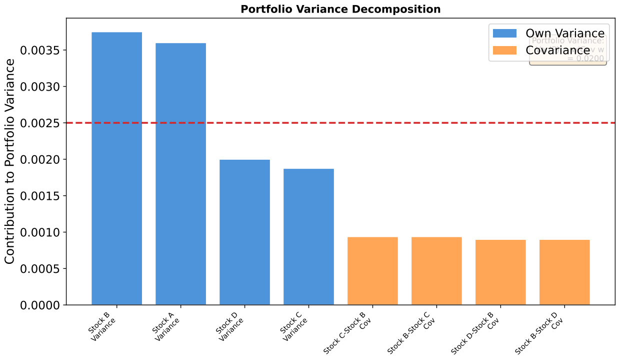
Left (negative) skew: long left tail, mean ¡ median.

Heavy-Tailed Distribution



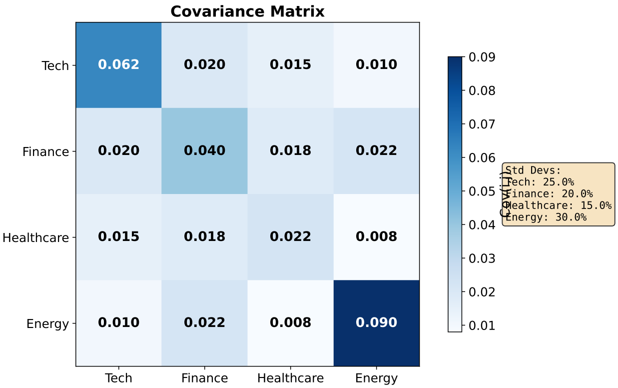
High kurtosis: extreme values more likely than normal.

Portfolio Variance Decomposition



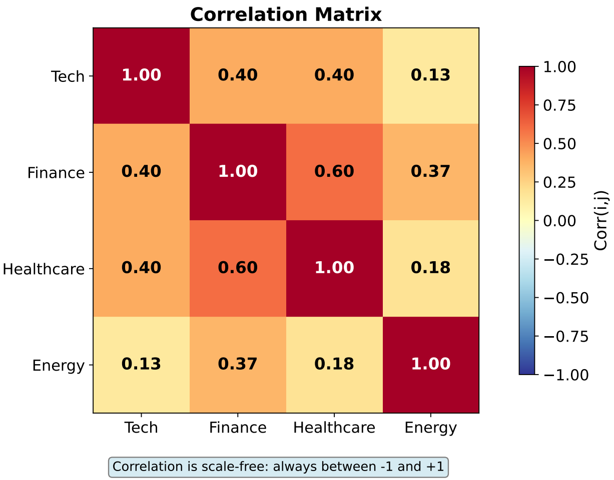
Portfolio variance comes from individual variances and covariances.

Covariance Matrix



Covariance matrix is symmetric; diagonal contains variances.

Correlation Matrix



Correlation standardizes covariance: always between -1 and +1.