Joint Distributions

Lesson 07

Digital Finance

Outline

Why Joint Distributions?

Multiple random variables together:

- Stock A and Stock B returns
- Interest rates and inflation
- Portfolio of multiple assets

Questions we want to answer:

- How do variables move together?
- What is the probability both exceed thresholds?
- How does one variable predict another?

Joint distributions capture relationships between variables.

Joint PMF (Discrete)

Joint PMF for discrete X and Y:

$$p(x, y) = P(X = x \text{ and } Y = y)$$

Properties:

- $p(x, y) \geq 0$
- $\sum_x \sum_y p(x, y) = 1$

Assigns probability to each combination of values.

Joint PDF (Continuous)

Think of it as a 3D surface:

- x-axis: values of X, y-axis: values of Y
- Height $f(x, y)$ = density at each point
- Probability = volume under the surface over a region

Like slicing a cake: $P(a \leq X \leq b, c \leq Y \leq d)$ is the volume of that slice.
Rules: Surface is never negative. Total volume = 1 (100%).

2D version of "area under the curve" – now it's volume under a surface.

Marginal Distributions

"Marginal" = focus on one variable, ignore the other.
How to find it:

- Discrete: Add up probabilities across all values of the other variable
- Continuous: Add up (integrate) over all values of the other variable

Visual intuition: Given a joint table, the "marginal" is literally the row totals (margin) or column totals.
Example: From $P(X = 1, Y = 1)$, $P(X = 1, Y = 2)$, ... sum to get $P(X = 1)$.

Marginal = "collapse" the joint distribution down to one variable.

Conditional Distributions

Distribution of Y given $X = x$:

Discrete:

$$p(y|x) = \frac{p(x,y)}{p_X(x)}$$

Continuous:

$$f(y|x) = \frac{f(x,y)}{f_X(x)}$$

Use: How does Y behave once we know X?

Conditional distribution updates beliefs about one variable given another.

Independence of Random Variables

X and Y are independent if:

$$f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{for all } x, y$$

Equivalent conditions:

- $f(y|x) = f_Y(y)$ (knowing X doesn't help predict Y)
- $P(X \leq a, Y \leq b) = P(X \leq a) \cdot P(Y \leq b)$

Finance: Asset returns are often NOT independent!

Independence simplifies calculations enormously.

Bivariate Normal Distribution

Two normal variables that move together (or opposite).
You specify 5 numbers:

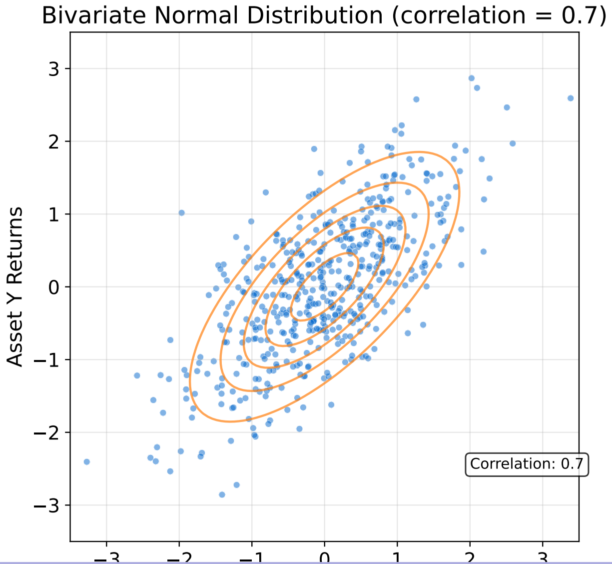
- μ_X, μ_Y = means of each variable
- σ_X, σ_Y = standard deviations of each
- ρ (rho) = correlation: how much they move together

Key properties:

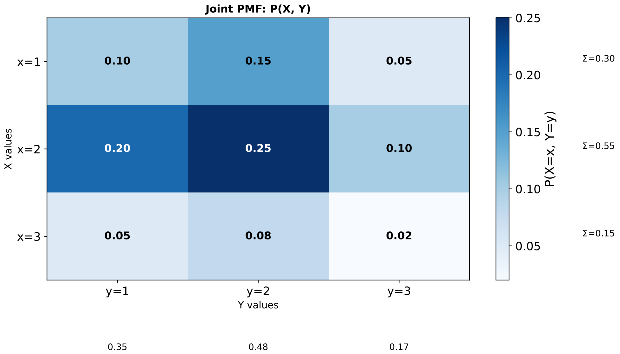
- Each variable alone is normal (marginals)
- If you fix one variable, the other is still normal (conditionals)
- If $\rho = 0$, they're independent

The "ellipse" tilts based on correlation. Foundation for portfolio theory.

Bivariate Normal: Visualization

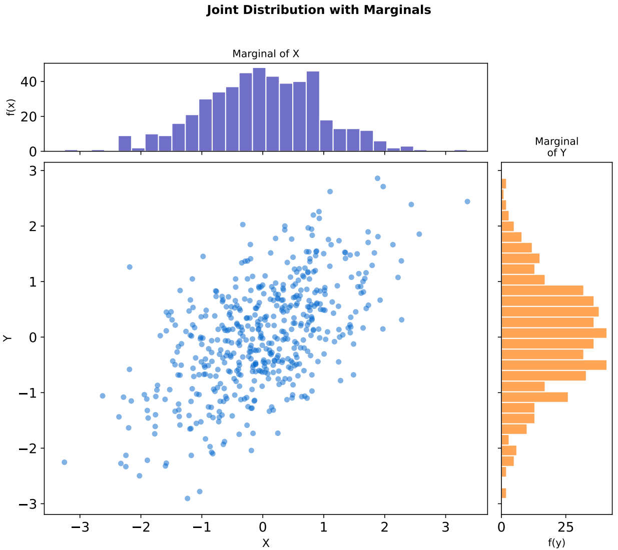


Joint PMF Visualization

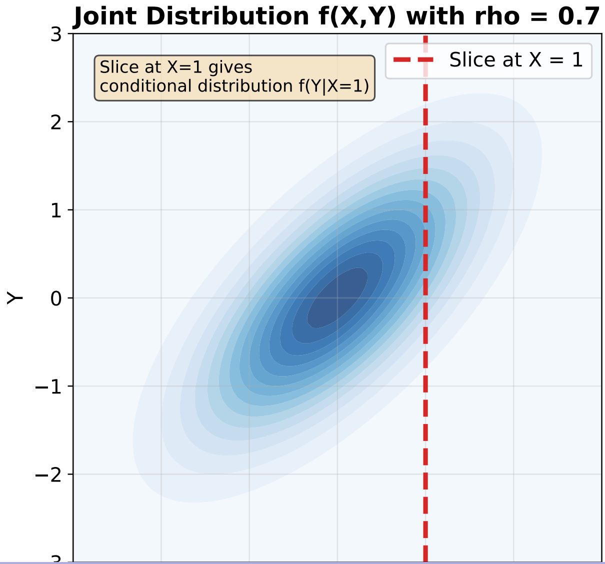


Joint distribution shows probabilities for pairs.

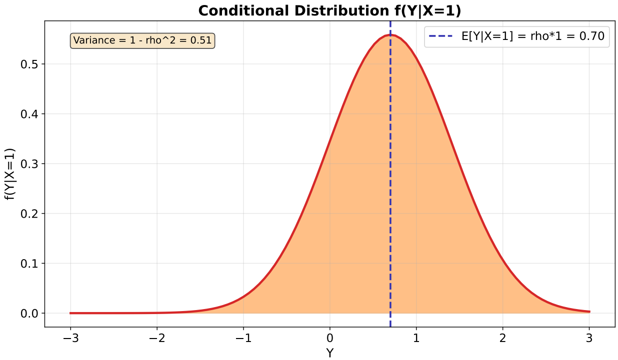
Marginal Distributions



Joint Distribution with Slice



Conditional Distribution f(Y—X=1)



Given X=1, Y is normally distributed with shifted mean.

Summary

Joint distributions:

- Describe multiple random variables together
- Joint $\rightarrow$ Marginal by integrating/summing out

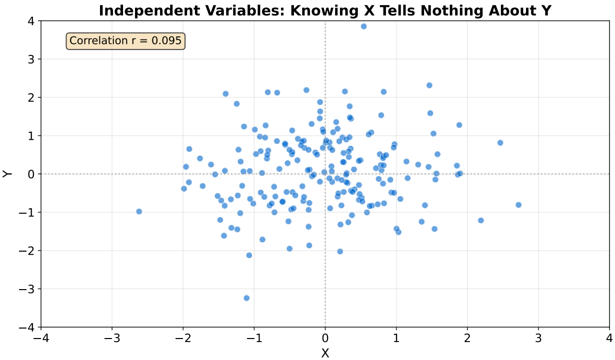
Conditional distributions:

- Distribution of one variable given the other
- Key for prediction and inference

Independence:

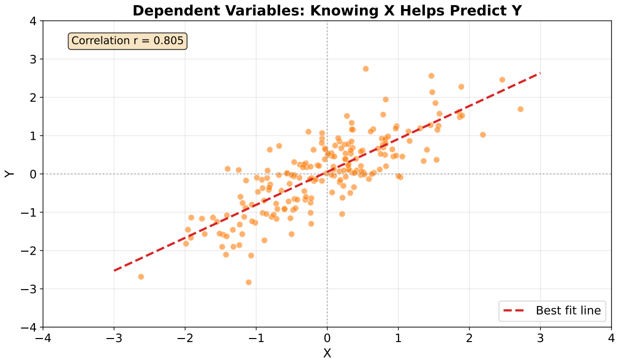
- Joint $=$ product of marginals
- Rarely holds perfectly in finance

Next lesson: Expectation and Moments



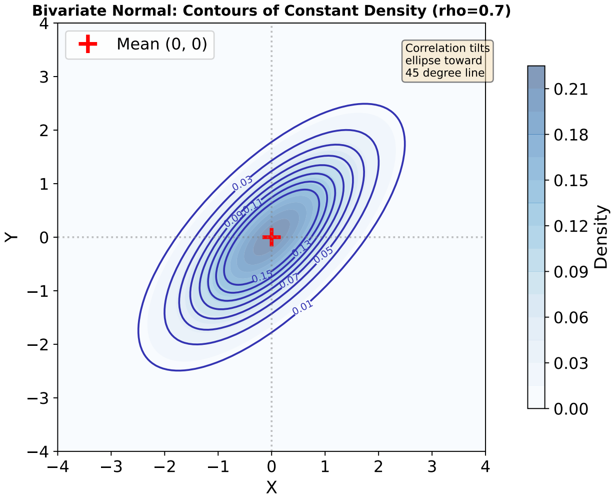
Independent: Knowing X tells you nothing about Y.

Dependent Variables



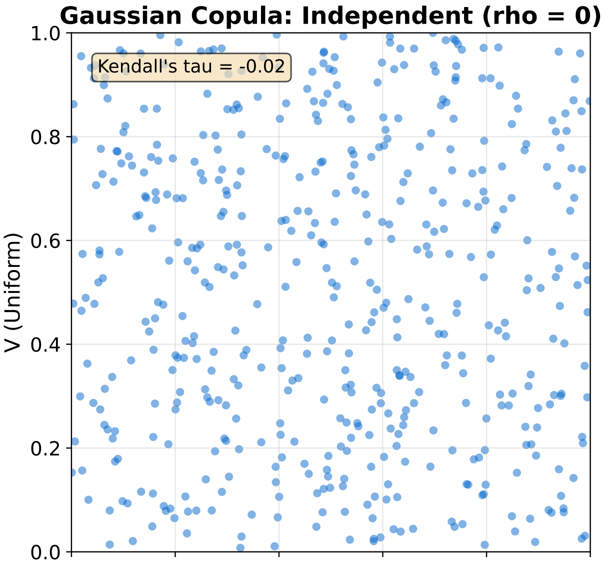
Dependent: Knowing X helps predict Y.

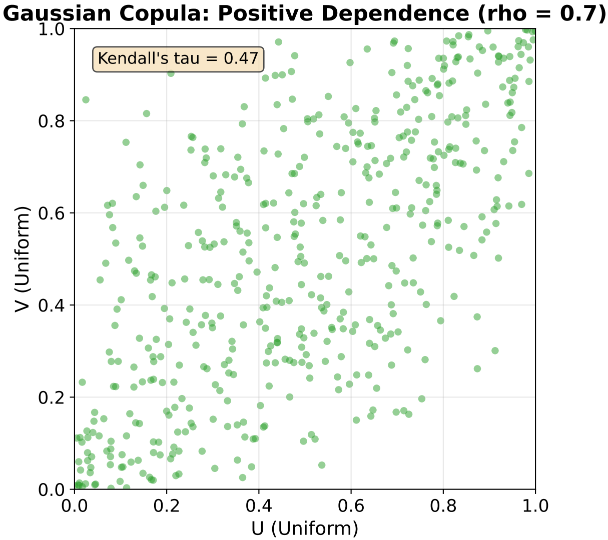
Bivariate Normal Contours



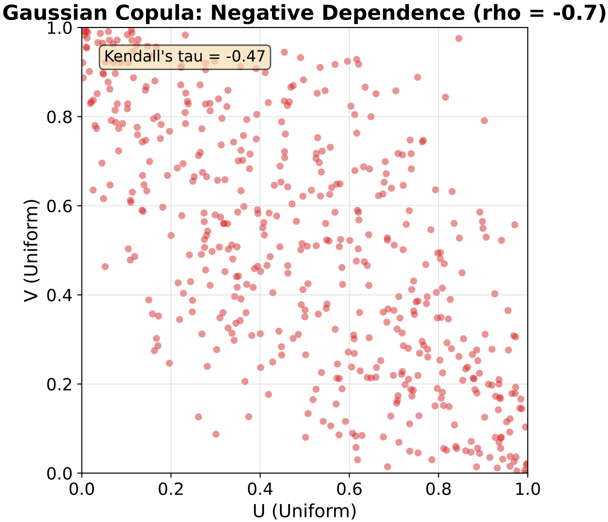
Contours show regions of equal probability density.

Copula: Independent Case



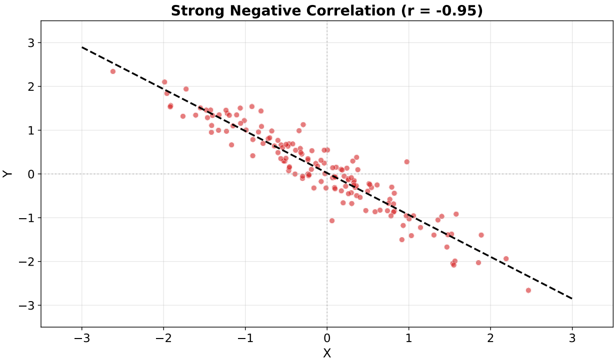

Gaussian Copula: Positive Dependence (rho = 0.7)
Kendall's tau = 0.47
V (Uniform)
U (Uniform)
0.0
0.2
0.4
0.6
0.8
1.0

Copula: Negative Dependence



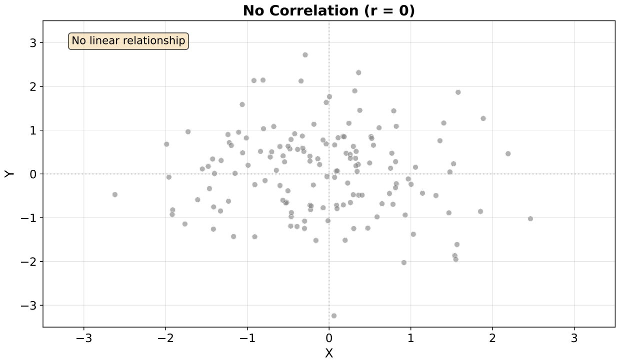
Points cluster along anti-diagonal: high U tends to come with low V.

Strong Negative Correlation



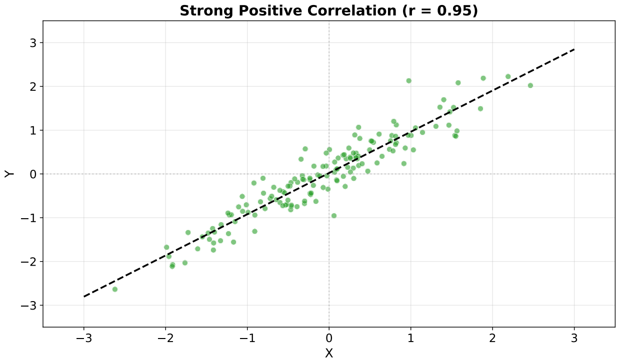
r near -1: As X increases, Y decreases predictably.

No Correlation



r near 0: No linear relationship between X and Y.

Strong Positive Correlation



r near +1: As X increases, Y increases predictably.