# Discrete Random Variables

## Lesson 05

Digital Finance

# Outline

# What is a Random Variable?

**Simple idea:** A random variable assigns a *number* to each random outcome.
**Example:** Roll two dice

- Outcome (3,4) $\rightarrow$ assign the number 7 (the sum)
- Outcome (1,1) $\rightarrow$ assign the number 2 (the sum)
- The variable $X =$ "sum of dice" can be 2, 3, ..., or 12

**Why useful?** Instead of tracking 36 outcomes, we track just 11 possible values of $X$.

**Notation:** Capital $X =$ random variable, lowercase $x =$ specific value.

**Random variable = rule that converts outcomes to numbers we can analyze.**

# Discrete vs Continuous

**Discrete random variable:**

- Takes countable values (finite or countably infinite)
- Examples: number of defaults, coin flips, dice rolls

**Continuous random variable:**

- Takes any value in an interval
- Examples: stock price, temperature, time

**This lesson:** Focus on discrete random variables

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**Continuous random variables covered in next lesson.**

# Probability Mass Function (PMF)

**PMF** gives probability for each value:

$$p(x) = P(X = x)$$

**Properties:**

- $p(x) \geq 0$ for all $x$
- $\sum_{\text{all } x} p(x) = 1$

**Example:** Fair die

$$p(x) = \frac{1}{6} \text{ for } x \in \{1, 2, 3, 4, 5, 6\}$$

**The PMF completely describes the distribution.**

# Cumulative Distribution Function (CDF)

**CDF** gives probability of being at most $x$:

$$F(x) = P(X \leq x) = \sum_{t \leq x} p(t)$$

**Properties:**

- $0 \leq F(x) \leq 1$
- Non-decreasing: if $a < b$, then $F(a) \leq F(b)$
- $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

**Useful for:** $P(a < X \leq b) = F(b) - F(a)$

**CDF accumulates probabilities up to each point.**

# Bernoulli Distribution

**Single trial with two outcomes:**

$$X \sim \text{Bernoulli}(p)$$

**PMF:**

$$p(x) = \begin{cases} p & x = 1 \text{ (success)} \\ 1 - p & x = 0 \text{ (failure)} \end{cases}$$

**Properties:**

- $E[X] = p$
- $\text{Var}(X) = p(1 - p)$

**Example:** Stock goes up (success) or down (failure)

**Building block for binomial distribution.**

# Binomial Distribution

**Number of successes in $n$ independent trials:**

$$X \sim \text{Binomial}(n, p)$$

**PMF:**

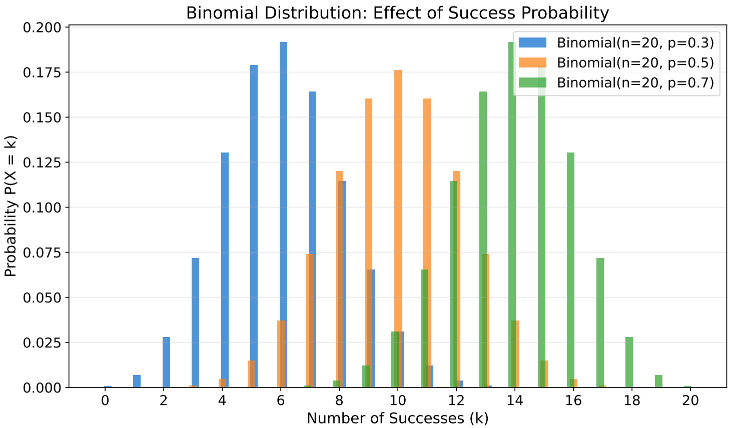
$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n$$

**Properties:**

- $E[X] = np$
- $\text{Var}(X) = np(1 - p)$

**Sum of $n$ independent Bernoulli random variables.**

# Binomial Distribution: Visualization




**Higher $p$ shifts the distribution right; symmetric at $p = 0.5$.**

# Poisson Distribution

**Number of events in fixed interval:**

$$X \sim \text{Poisson}(\lambda)$$

**PMF:**

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

**Properties:**

- $E[X] = \lambda$
- $\text{Var}(X) = \lambda$

**Examples:** Defaults per year, trades per minute, claims per month

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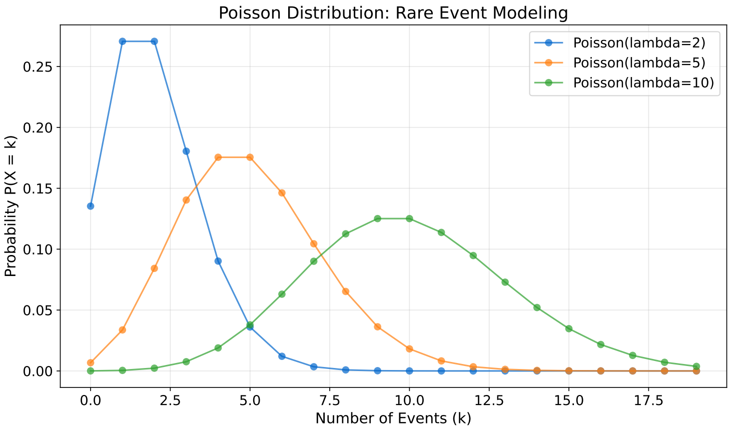
**Used for counting rare events in a fixed period/region.**




**As $\lambda$ increases, distribution spreads and becomes more symmetric.**

# Geometric Distribution

**Question:** How many tries until first success?
$X \sim \text{Geometric}(p)$ where $p =$ probability of success on each try.

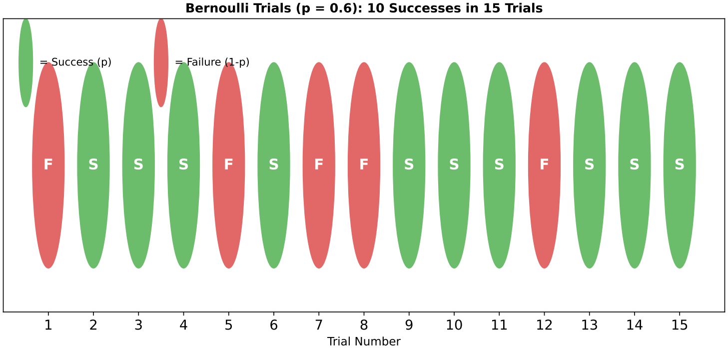
$$P(X = k) = (1 - p)^{k-1} \cdot p \quad \text{(fail } k - 1 \text{ times, then succeed)}$$

**Key insight – Memoryless property:**

- "I've failed 10 times. Am I due for a success?" No!
- Past failures don't change future probability
- Each trial is fresh, like the coin has no memory

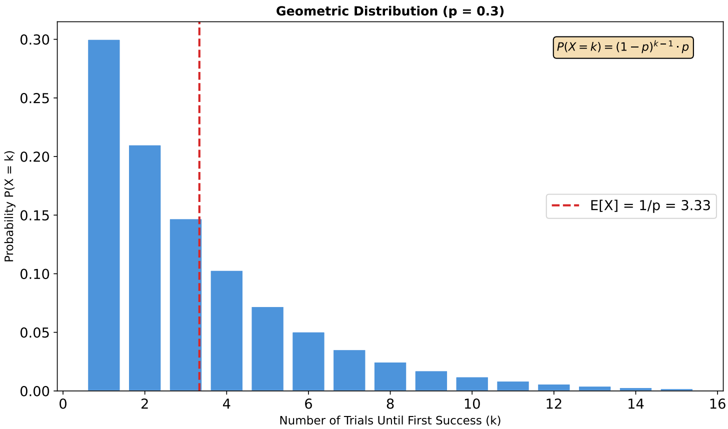
**Mean = $1/p$ tries. If $p = 0.1$, expect 10 tries on average.**

# Sequence of Bernoulli Trials



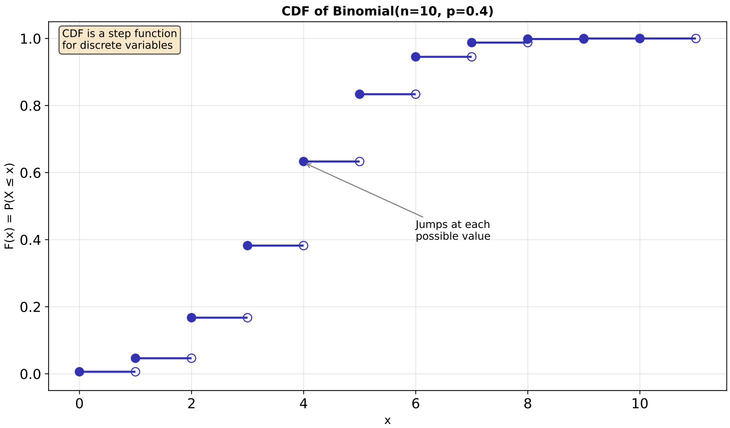

**Each trial is independent with fixed success probability.**

# Geometric Distribution




**Models waiting time until first success.**

# CDF: Cumulative Distribution



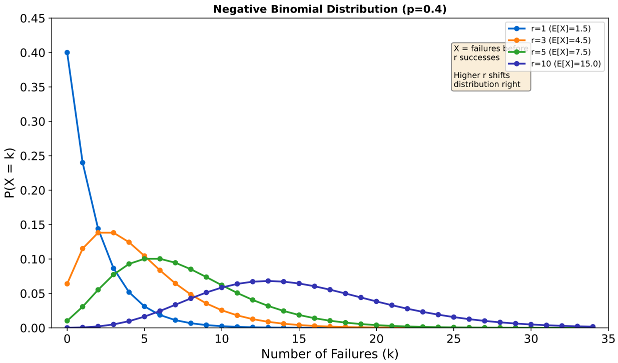

**CDF shows probability of being at or below each value.**

# Distribution Summary

| Distribution | Parameters | Mean | Variance |
|---|---|---|---|
| Bernoulli | $p$ | $p$ | $p(1-p)$ |
| Binomial | $n, p$ | $np$ | $np(1-p)$ |
| Poisson | $\lambda$ | $\lambda$ | $\lambda$ |
| Geometric | $p$ | $1/p$ | $(1-p)/p^2$ |

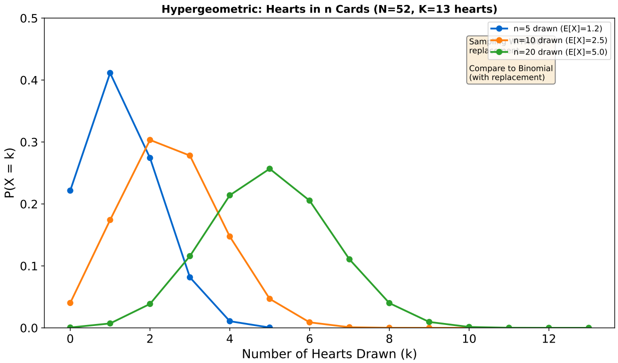
**Next lesson: Continuous Random Variables**

# Negative Binomial Distribution



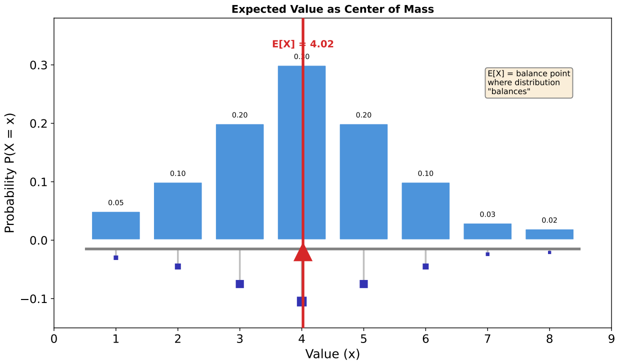

**PMF:** $P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$ **for** $k = r, r+1, \ldots$; **E[X] = r/p.**

# Hypergeometric Distribution



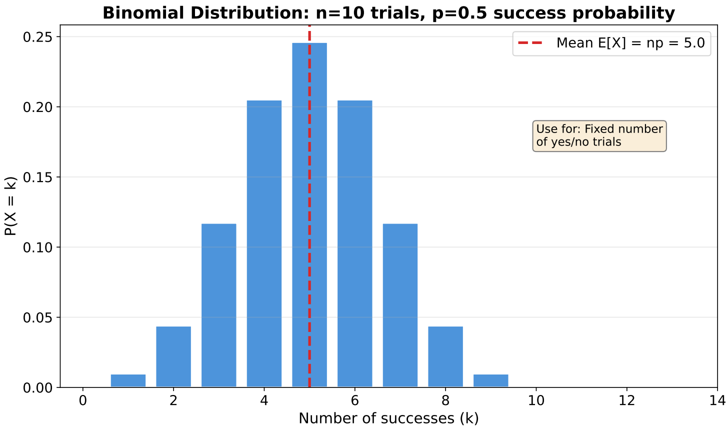

**PMF:** $P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$ **; sampling without replacement.**

# Expected Value as Balance Point



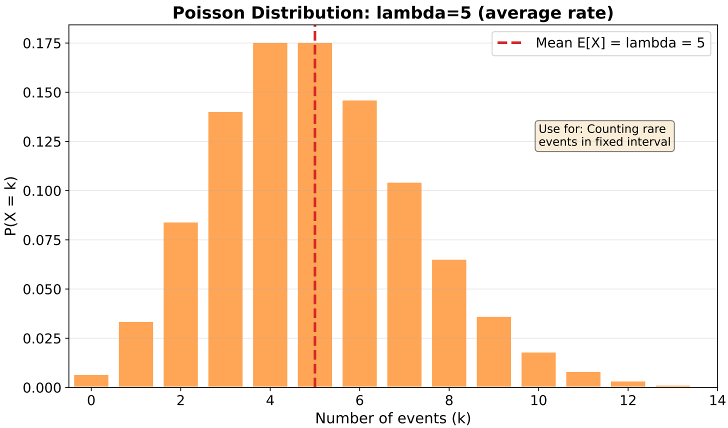

**The mean is the center of mass of the distribution.**

# Binomial Distribution



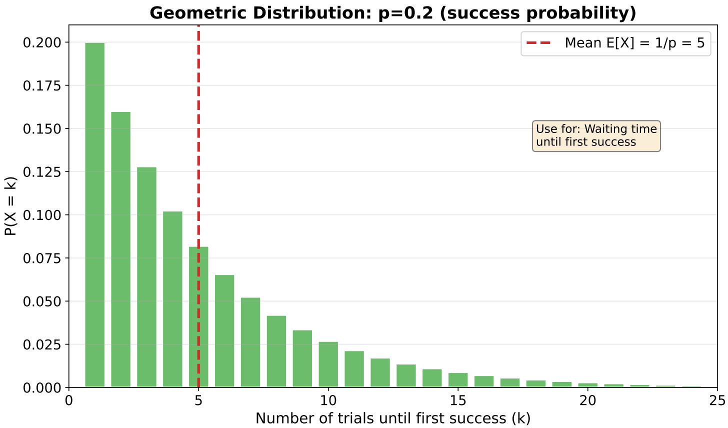

**Binomial: Fixed n trials, counting successes.**

# Poisson Distribution




**Poisson: Counting rare events in fixed interval (no upper bound).**

# Geometric Distribution




**Geometric: Waiting time until first success.**