Set Theory and Probability Axioms

Lesson 03

Digital Finance

Outline

What is a Set?

Set: A collection of distinct objects (think: a bag of things)
Example: $A = \{1, 2, 3, 4, 5\}$ is a set of five numbers.

- The number 3 is "in" the set: we write $3 \in A$ ("3 is in A")
- The number 7 is "not in" the set: we write $7 \notin A$ ("7 is not in A")

Reading the symbols:

- $\in$ means "is an element of" or "is in"
- $\notin$ means "is not an element of" or "is not in"

Sets are like labeled boxes – we care what's inside, not the order.

Special Sets

Empty set $\emptyset$: Contains no elements

- Also written as $\{\}$
- Example: Set of stock prices less than zero

Universal set U or S: Contains all possible elements

- In probability, called the **sample space**
- Example: All possible outcomes of rolling a die

Subset $A \subseteq B$: Every element of A is in B

- $A \subset B$: A is a proper subset (not equal to B)

Every set is a subset of itself; empty set is subset of every set.

Set Operations

Four ways to combine sets (see Venn diagram on next slide):

Union $A \cup B$: Everything in A **or** B (or both)

- Think: "A or B" – combine both sets

Intersection $A \cap B$: Only what's in **both** A **and** B

- Think: "A and B" – the overlap

Complement A^c: Everything **not** in A

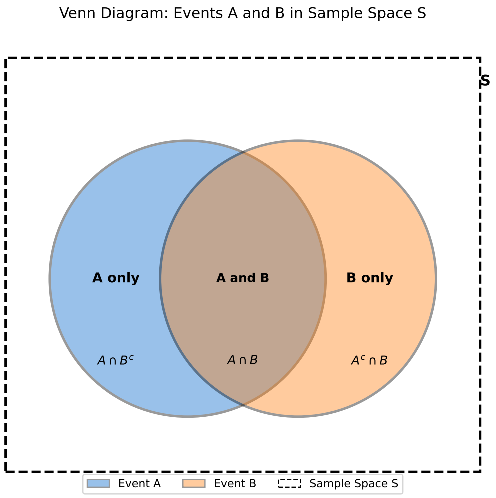
- Think: "not A" – everything outside A

Difference $A \setminus B$: In A but **not** in B

- Think: "A minus B" – remove B from A

These operations match logic: "or" = $\cup$, "and" = $\cap$, "not" = complement.

Venn Diagrams



Venn diagrams visualize set relationships and operations.

De Morgan's Laws (most important!):
In plain English:

- "NOT (A or B)" = "(NOT A) and (NOT B)"
- "NOT (A and B)" = "(NOT A) or (NOT B)"

In symbols: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

Example: "NOT (rain or snow)" = "no rain AND no snow"

Other useful properties:

- Order doesn't matter: $A \cup B = B \cup A$
- Grouping doesn't matter: $A \cup (B \cup C) = (A \cup B) \cup C$

De Morgan's Laws flip "and" to "or" when you take the complement.

Sample Space

Random experiment: Process with uncertain outcome

- Rolling a die, flipping a coin, stock price change

Sample space S: Set of all possible outcomes

- Coin flip: $S = \{\text{H}, \text{T}\}$
- Die roll: $S = \{1, 2, 3, 4, 5, 6\}$
- Stock return: $S = \mathbb{R}$ (all real numbers)

Types of sample spaces:

- **Discrete**: Countable outcomes (finite or countably infinite)
- **Continuous**: Uncountable outcomes (interval of real numbers)

The sample space defines what outcomes are possible.

Events

Event: Any subset of the sample space

- **Simple event**: Single outcome
- **Compound event**: Multiple outcomes

Examples (die roll):

- Event $A =$ "roll an even number" $= \{2, 4, 6\}$
- Event $B =$ "roll greater than 4" $= \{5, 6\}$
- $A \cap B =$ "even AND greater than 4" $= \{6\}$
- $A \cup B =$ "even OR greater than 4" $= \{2, 4, 5, 6\}$

Special events:

- S: Certain event (always occurs)
- $\emptyset$: Impossible event (never occurs)

Events are what we assign probabilities to.

Mutually Exclusive Events

Mutually exclusive (disjoint): Events cannot occur simultaneously

$$A \cap B = \emptyset$$

Examples:

- Coin flip: Heads and Tails are mutually exclusive
- Die roll: "odd" and "even" are mutually exclusive

Exhaustive events: Together cover entire sample space

$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$

Partition: Events that are both mutually exclusive AND exhaustive

Partitions divide the sample space into non-overlapping pieces.

What is Probability?

Probability measures the likelihood of an event occurring

Three interpretations:

- **Classical**: Ratio of favorable to total outcomes
- **Frequentist**: Long-run relative frequency
- **Subjective**: Degree of belief

Finance applications:

- Probability of default: 2%
- Probability stock rises tomorrow: 52%
- Probability of recession: 30%

Probability provides a rigorous language for uncertainty.

Kolmogorov's Axioms

Three simple rules that all probabilities must follow:

Rule 1: Probabilities are never negative

- $P(A) \geq 0$ for any event A

Rule 2: Something must happen

- $P(\text{sample space}) = 1$ (certainty)

Rule 3: For non-overlapping events, add the probabilities

- If A and B can't happen together: $P(A \text{ or } B) = P(A) + P(B)$

That's it! Everything else in probability follows from these three rules.

These are Kolmogorov's axioms (1933). Simple rules, profound consequences.

Properties from Axioms

Derived from the axioms:

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$
- $0 \leq P(A) \leq 1$ for all events A
- If $A \subseteq B$, then $P(A) \leq P(B)$

Addition rule (general):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Why subtract? To avoid double-counting the intersection.

These properties are consequences, not assumptions.

Calculating Probabilities

For equally likely outcomes (classical approach):

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Total number of outcomes}} = \frac{|A|}{|S|}$$

Example: Rolling a fair die

- $P(\text{even}) = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = 0.5$
- $P(\text{greater than 4}) = \frac{|\{5,6\}|}{6} = \frac{2}{6} = \frac{1}{3}$

Key assumption: All outcomes must be equally likely!

This formula only works when outcomes are equally probable.

Why Counting Matters

To use the classical formula, we need to count:

- How many outcomes are in the event?
- How many outcomes are in the sample space?

Finance examples:

- How many 5-stock portfolios from 100 stocks?
- How many ways to rank 10 investment options?
- How many passwords of length 8?

Counting gets complex quickly!

- We need systematic counting techniques

Counting is the key to computing classical probabilities.

Multiplication Principle

If task has k stages with n_i choices at stage i:

$$\text{Total ways} = n_1 \times n_2 \times \cdots \times n_k$$

Example: Creating a password

- 3 letters (26 choices each) + 2 digits (10 choices each)
- Total: $26 \times 26 \times 26 \times 10 \times 10 = 1,757,600$

Finance example: Portfolio construction

- Choose 1 stock from sector A (5 options)
- Choose 1 stock from sector B (8 options)
- Choose 1 bond (3 options)
- Total portfolios: $5 \times 8 \times 3 = 120$

The multiplication principle is the foundation of counting.

Permutations

Permutation: Ordered arrangement (order matters!)
First, what is factorial? $n! = n \times (n-1) \times \cdots \times 2 \times 1$

- Example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- Special: $0! = 1$ (by definition)

Choosing and arranging k items from n:

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1)$$

Example: Ranking top 3 from 10 stocks

- First place: 10 choices. Second: 9 left. Third: 8 left.
- $P(10,3) = 10 \times 9 \times 8 = 720$ ways

ABC, ACB, BAC are different permutations – order matters!

Combinations

Combination: Selection where order doesn't matter
Key insight: Combinations = Permutations ÷ (ways to rearrange)

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} \quad \text{(read: ``n choose k'')}$$

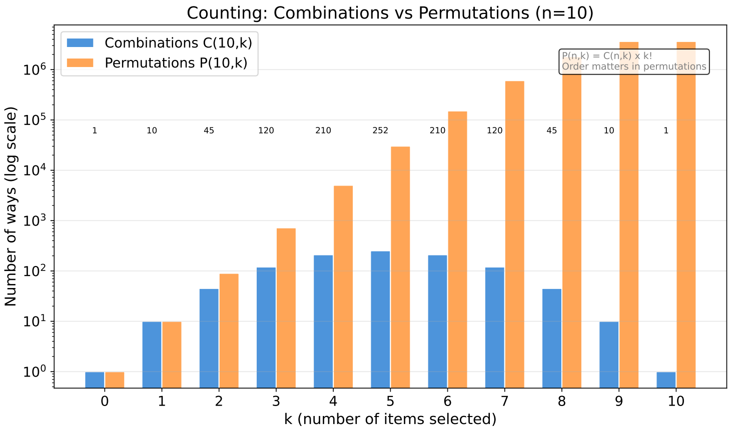
Example: Selecting 3 stocks from 10 for a portfolio

- Permutations: $10 \times 9 \times 8 = 720$
- But {A,B,C}, {A,C,B}, {B,A,C}, etc. are the same portfolio!
- Divide by $3! = 6$ rearrangements: $720 \div 6 = 120$ combinations

When to use: "Choose 3 stocks" (combination) vs "Rank top 3" (permutation)

Combinations ¡ Permutations because order doesn't create new selections.

Permutations vs Combinations



Permutations $\geq$ Combinations because order multiplies possibilities.

Counting Summary

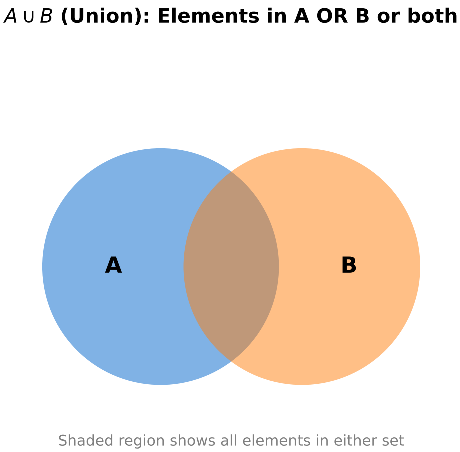
Scenario	Formula	Example
Arrangements of n	$n!$	Arrange 5 books
Ordered selection k from n	$P(n,k) = \frac{n!}{(n-k)!}$	Rank top 3 of 10
Unordered selection k from n	$C(n,k) = \frac{n!}{k!(n-k)!}$	Choose 3 of 10
With replacement	n^k	k dice rolls

Decision tree:

- Does order matter? Yes $\rightarrow$ Permutation
- Does order matter? No $\rightarrow$ Combination

Ask: "Would rearranging give a different outcome?"

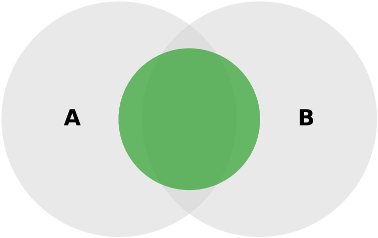
Union of Sets



Union combines elements from both sets (A OR B).

Intersection of Sets

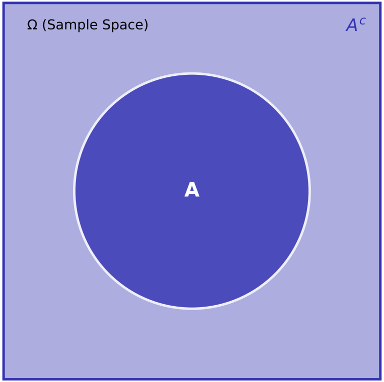
$A \cap B$ (Intersection): Elements in BOTH A and B



Green region shows elements common to both sets

Intersection finds elements common to both sets (A AND B).

Complement of an Event

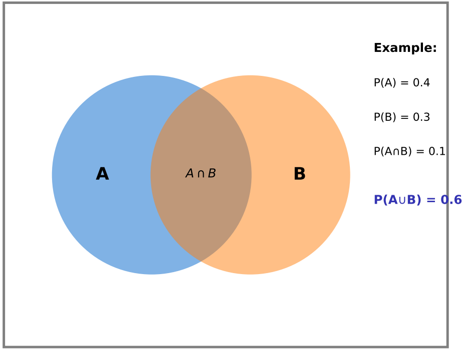


A^c = Complement of A = Everything NOT in A

$P(A) + P(A^c) = 1$

$P(A^c) = 1 - P(A)$: **What doesn't happen.**

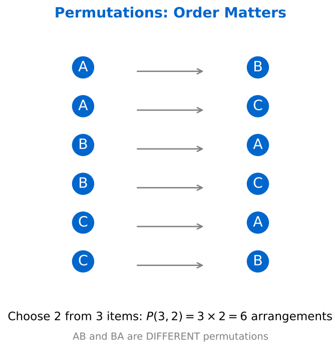
Addition Rule



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

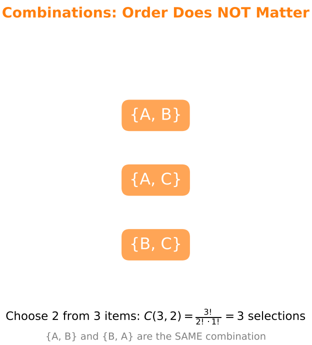
P(A or B) = P(A) + P(B) - P(A and B).

Permutations: Order Matters



P(n,k) counts arrangements where AB and BA are different.

Combinations: Order Does Not Matter



C(n,k) counts selections where {A,B} and {B,A} are the same.

Summary

Set theory provides the language:

- Sample space, events, operations (union, intersection, complement)

Probability axioms provide the rules:

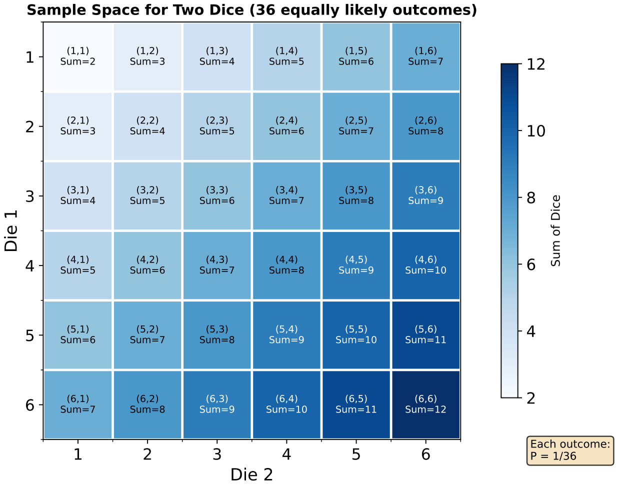
- Non-negativity, normalization, additivity
- All properties derived from these three axioms

Counting provides the tools:

- Permutations (order matters)
- Combinations (order doesn't matter)

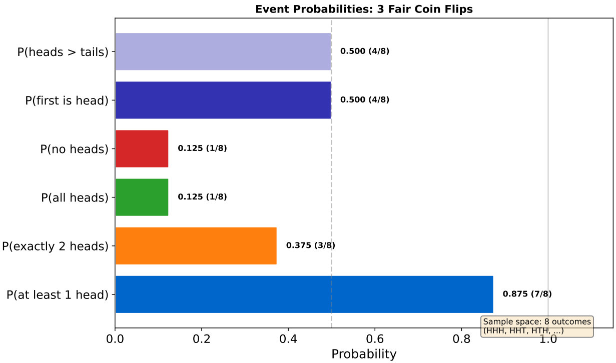
Next lesson: Conditional Probability and Bayes' Theorem

Sample Space: Two Dice



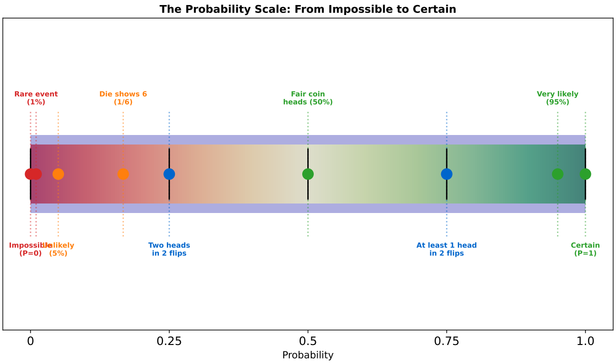
The 6x6 grid shows all 36 equally likely outcomes.

Event Probabilities



Different events have different probabilities based on favorable outcomes.

Probability Scale



Probability ranges from 0 (impossible) to 1 (certain).