Linear Algebra Week 5

Eigenvalues, Eigenvectors, and Applications

Topics: Complex numbers, eigenvalue theory, diagonalization, Markov chains, PageRank

Overview: Week 5 Topics

Part I: Foundations

- Complex numbers and arithmetic
- Similar matrices and eigenvalue preservation

Part II: Eigenvalue Theory

- Eigenvalues and eigenvectors: definition and computation
- Algebraic vs geometric multiplicity
- Characteristic polynomial and eigenspaces

Part III: Diagonalization

- Diagonalizability conditions
- Diagonalization algorithm and applications
- Spectral decomposition

Part IV: Applications

- Markov chains and stochastic matrices
- PageRank algorithm

Week 5 combines theory with powerful real-world applications

Beginner Level: Introduction to Complex Numbers

Real Numbers Are Not Enough!
Consider the equation:

$$x^2 = -1$$

- In real numbers: No solution! (Squaring always gives positive)
- We need to extend the number system

Solution: Introduce the Imaginary Unit

$$i = \sqrt{-1}$$

Key property:

$$\boxed{i^2 = -1}$$

Complex Number:

$$z = a + bi$$

where a is the *real part*, b is the *imaginary part*

Complex numbers extend real numbers by adding imaginary unit i where i squared equals -1

The Complex Plane: Visualizing Complex Numbers

How to Draw Complex Numbers?

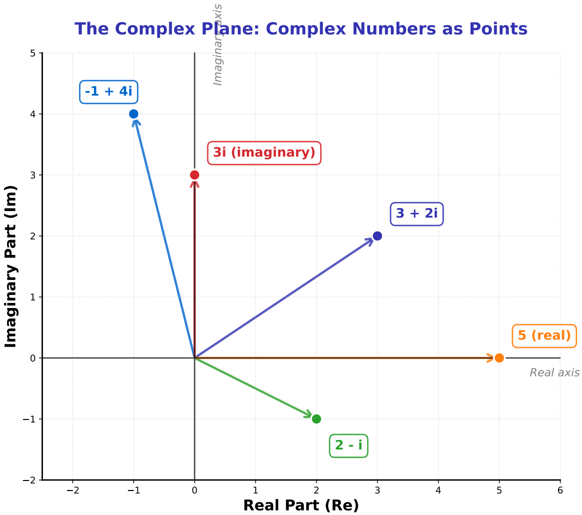
Every complex number $z = a + bi$ is a point in 2D:

- Horizontal axis: Real part (a)
- Vertical axis: Imaginary part (b)

Examples:

- $3 + 2i \rightarrow$ Point at (3, 2)
- $-1 + 4i \rightarrow$ Point at (-1, 4)
- $2 - i \rightarrow$ Point at (2, -1)
- $5 \rightarrow$ Point at (5, 0) - pure real
- $3i \rightarrow$ Point at (0, 3) - pure imaginary

Complex numbers are points or vectors in the 2D complex plane



Complex numbers are points or vectors in the 2D complex plane

Adding and Subtracting Complex Numbers

Simple Rule: Combine Like Terms

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Example 1:

$$(3 + 2i) + (1 + 4i) = (3 + 1) + (2 + 4)i = 4 + 6i$$

Example 2:

$$(5 + i) - (2 + 3i) = (5 - 2) + (1 - 3)i = 3 - 2i$$

Geometric Interpretation:

- Addition: Vector addition (tip-to-tail)
- Subtraction: Vector from second to first

Addition and subtraction are component-wise, just like 2D vectors

Why Do We Need Complex Numbers?

1. Solving Polynomial Equations
$x^2 + 1 = 0$ has no real solutions, but:

$$x = i \text{ and } x = -i$$

2. Fundamental Theorem of Algebra
Every polynomial of degree n has exactly n complex roots (counting multiplicity)

3. Applications Everywhere:

- **Electrical Engineering**: AC circuits, impedance
- **Signal Processing**: Fourier transforms, filters
- **Quantum Mechanics**: Wave functions
- **Control Systems**: Stability analysis

Complex numbers complete the number system and are essential for many applications

Introduction: Matrices That Look Different But Act the Same

Motivating Question:
Can two different-looking matrices represent the same transformation?

Example:
Consider rotation by 90 degrees:

- In standard basis: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- In different basis: $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (over complex numbers)

They look different but represent the same geometric operation!

This is the idea of Similar Matrices
Same transformation, different coordinate systems

Similar matrices are different representations of the same linear transformation

BSc Level: Complex Arithmetic and Similar Matrices

Multiplying Complex Numbers

Use Distributive Law and $i^2 = -1$

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= ac + (ad+bc)i + bd(-1) \\ &= (ac-bd) + (ad+bc)i\end{aligned}$$

Example:

$$\begin{aligned}(3+2i)(1+4i) &= 3(1) - 2(4) + [3(4) + 2(1)]i \\ &= 3 - 8 + [12+2]i = -5 + 14i\end{aligned}$$

Verification:

$$\begin{aligned}(3+2i)(1+4i) &= 3 + 12i + 2i + 8i^2 \\ &= 3 + 14i - 8 = -5 + 14i\end{aligned}$$

Multiplication follows FOIL method with the rule that i squared equals -1

Complex Conjugate and Division

Complex Conjugate:
For $z = a + bi$, the conjugate is:

$$\overline{z} = a - bi$$

Key Properties:

- $z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2$ (always real and positive!)
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Division:
Multiply numerator and denominator by conjugate of denominator:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

The conjugate is crucial for division and finding modulus of complex numbers

Modulus (Absolute Value):
For $z = a + bi$:

$$|z| = \sqrt{a^2 + b^2}$$

This is the distance from origin in complex plane.

Polar Form:
Every complex number can be written as:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where:

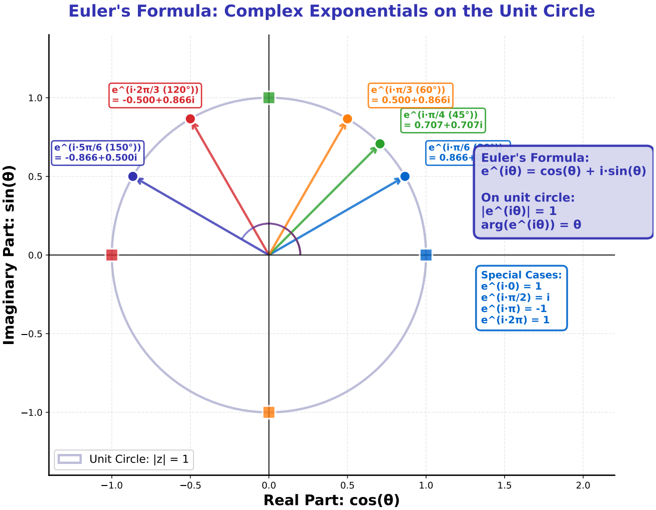
- $r = |z| = \sqrt{a^2 + b^2}$ (modulus)
- $\theta = \arg(z) = \arctan(b/a)$ (argument/angle)

Euler's Formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Polar form expresses complex numbers using magnitude and angle instead of real and imaginary parts

Euler's Formula: Unit Circle



Euler's formula connects complex exponentials to trigonometry on the unit circle

Multiplication in Polar Form

The Magic of Polar Form:
If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Rule:

- Multiply moduli: $r_1 \cdot r_2$
- Add angles: $\theta_1 + \theta_2$

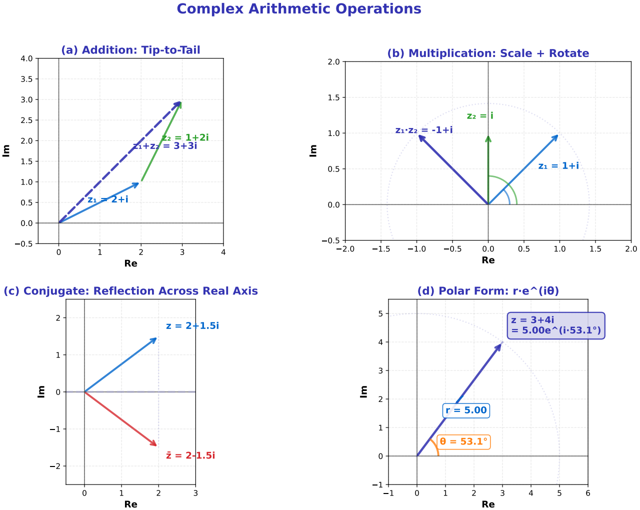
Example:
$z_1 = 2e^{i\pi/4}$ and $z_2 = 3e^{i\pi/6}$

$$z_1 \cdot z_2 = 6e^{i(\pi/4+\pi/6)} = 6e^{i5\pi/12}$$

Geometric Meaning:
Multiplication = scaling + rotation!

Polar form makes multiplication geometrically intuitive: scale and rotate

Complex Arithmetic: Operations Visualized



Four fundamental operations: addition, multiplication, conjugate, and polar form

Definition
Matrices $A, B \in \mathbb{R}^{n\times n}$ are **similar** if there exists an invertible matrix P such that:

$$\boxed{B = P^{-1}AP}$$

We write: $A \sim B$

Interpretation:

- A and B represent the same linear transformation
- But in different coordinate systems (bases)
- P is the change-of-basis matrix

Equivalence Relation:

- Reflexive: $A \sim A$ (take $P = I$)
- Symmetric: If $A \sim B$ then $B \sim A$
- Transitive: If $A \sim B$ and $B \sim C$ then $A \sim C$

Similar matrices represent the same transformation in different coordinate systems

Similarity Preserves Eigenvalues

Key Theorem:
If $A \sim B$ (i.e., $B = P^{-1}AP$), then A and B have the same eigenvalues.

Intuitive Reason:
Eigenvalues describe *how much* a transformation stretches in certain directions. Changing coordinates does not change the stretching factors!

Proof Sketch:
If $Av = \lambda v$, then:

$$\begin{aligned} B(P^{-1}v) &= P^{-1}AP(P^{-1}v) \\ &= P^{-1}Av \\ &= P^{-1}(\lambda v) \\ &= \lambda(P^{-1}v) \end{aligned}$$

So $P^{-1}v$ is an eigenvector of B with eigenvalue λ.

Eigenvalues are invariant under similarity transformations

Example 1: Rotation Matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is similar to its diagonal form (over complex numbers):

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

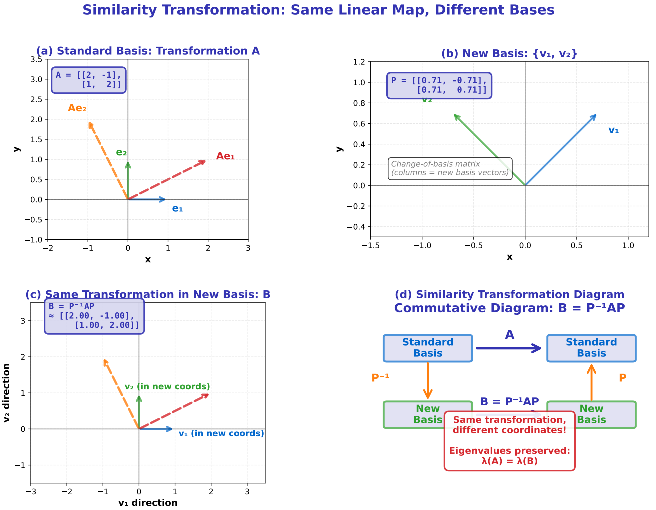
Both represent 90-degree rotation, eigenvalues: $i, -i$

Example 2: Diagonal Matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \sim B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Related by permutation matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Similar matrices can look very different but share fundamental properties



Similarity connects different representations of the same linear transformation

Exercise 1: Complex Number Calculations

Calculate the following:

(a) $(2 + 3i) + (1 - 4i)$
(b) $(1 + i)(1 - i)$
(c) $\frac{3+4i}{1+2i}$
(d) Express $z = 1 + i$ in polar form
(e) Calculate $(1 + i)^2$

Hints:

- Use $i^2 = -1$
- For division, multiply by conjugate
- For polar: $r = |z|$, $\theta = \arctan(b/a)$

Practice these fundamental operations to build fluency with complex numbers

Exercise 2: Checking Similarity

Problem:

Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ are similar.

Steps:

1. Find a matrix P such that $B = P^{-1}AP$
2. Verify by computation
3. Check that both matrices have the same eigenvalues

Hint: Try $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Verifying similarity requires finding the change-of-basis matrix

Advanced Level: Complex Eigenvalues and Similarity Properties

Real Matrices Can Have Complex Eigenvalues!
Consider:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

Eigenvalues: $\lambda_1 = i$, $\lambda_2 = -i$

Complex Eigenvectors:
For $\lambda = i$:

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

Solution: $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ or $v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Complex eigenvalues appear in conjugate pairs for real matrices

Complex Conjugate Pairs

Theorem:
If A is a real matrix and $\lambda = a + bi$ is a complex eigenvalue with eigenvector v, then:

- $\overline{\lambda} = a - bi$ is also an eigenvalue
- $\overline{v}$ is the corresponding eigenvector

Consequence:
Complex eigenvalues always come in conjugate pairs for real matrices!

Geometric Meaning:
Complex eigenvalues $\pm bi$ correspond to rotations (no real eigenvectors = no invariant directions)

Example:
Rotation matrix: eigenvalues $e^{i\theta}, e^{-i\theta}$ represent the rotation angle

Complex eigenvalues encode rotational behavior of linear transformations

Properties Preserved Under Similarity

If $A \sim B$ (i.e., $B = P^{-1}AP$), then:

1. **Eigenvalues:** $\lambda(A) = \lambda(B)$
2. **Trace:** $\mathrm{tr}(A) = \mathrm{tr}(B)$
3. **Determinant:** $\det(A) = \det(B)$
4. **Rank:** $\mathrm{rank}(A) = \mathrm{rank}(B)$
5. **Characteristic polynomial:** $p_A(\lambda) = p_B(\lambda)$
6. **Minimal polynomial:** $m_A(\lambda) = m_B(\lambda)$

Proof for Trace:

$$\begin{aligned} \mathrm{tr}(P^{-1}AP) &= \mathrm{tr}(APP^{-1}) \\ &= \mathrm{tr}(A) \end{aligned}$$

(using cyclic property of trace)

Many important matrix properties are invariant under similarity

Deep Interpretation:
Let A represent linear transformation T in basis $\mathcal{B}$, and B represent T in basis $\mathcal{B}'$.
If P is the change-of-basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, then:

$$B = P^{-1}AP$$

Workflow:

$$\begin{array}{ccc} \mathcal{B}'\text{-coordinates} & \xrightarrow{B} & \mathcal{B}'\text{-coordinates} \\ \downarrow P & & \uparrow P^{-1} \\ \mathcal{B}\text{-coordinates} & \xrightarrow{A} & \mathcal{B}\text{-coordinates} \end{array}$$

Key Insight:
Similarity transformations do not change the transformation, only the coordinate system we use to describe it!

Similarity captures the invariant essence of a linear transformation across bases

Diagonalization as Similarity

Definition:
A is **diagonalizable** if $A \sim D$ for some diagonal matrix D.

Theorem:
A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.
In this case:

$$A = PDP^{-1}$$

where:

- $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ (eigenvalues)
- $P = [v_1 | \cdots | v_n]$ (eigenvectors as columns)

Why Diagonalize?

- Simplifies matrix powers: $A^k = PD^kP^{-1}$
- Reveals eigenstructure explicitly
- Decouples systems of differential equations

Diagonalization is a special case of similarity transformation to the simplest form

Advanced Exercise 1: Complex Eigenvalues

Problem:
Find all eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Steps:

1. Compute characteristic polynomial $\det(A - \lambda I)$
2. Solve for eigenvalues (they will be complex!)
3. Find eigenvectors for each eigenvalue
4. Verify that eigenvalues are complex conjugates

Interpret: What geometric transformation does this matrix represent?

This exercise combines complex numbers with eigenvalue theory

Advanced Exercise 2: Similarity Invariants

Problem:
Given:

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

(a) Are A and B similar? Justify your answer using:

- Trace
- Determinant
- Eigenvalues
- Rank

(b) If they are similar, find the change-of-basis matrix P.
(c) If not, explain which property fails.

Use multiple similarity invariants to check whether matrices are similar

What if a matrix is not diagonalizable?
Not all matrices can be diagonalized, but they can be brought to **Jordan Normal Form**:

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & \cdots \\ 0 & \lambda_1 & 1 & \cdots \\ 0 & 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Jordan Blocks:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Jordan Normal Form is the canonical form for similarity up to permutation

Complex Numbers:

- Extend reals with imaginary unit i where $i^2 = -1$
- Arithmetic: addition, multiplication, conjugate, division
- Polar form: $z = re^{i\theta}$, Euler's formula
- Essential for solving polynomials and applications

Similar Matrices:

- $A \sim B$ if $B = P^{-1}AP$ for invertible P
- Represent same transformation in different bases
- Preserve: eigenvalues, trace, determinant, rank
- Diagonalization is special case of similarity

Connection:
Complex eigenvalues arise naturally from real matrices, making complex numbers essential for understanding linear transformations!

These concepts form the foundation for eigenvalue theory in Week 5 Part 2

Beginner Level: Introduction to Eigenvalues

What is an Eigenvector?

The fundamental equation

$$Av = \lambda v$$

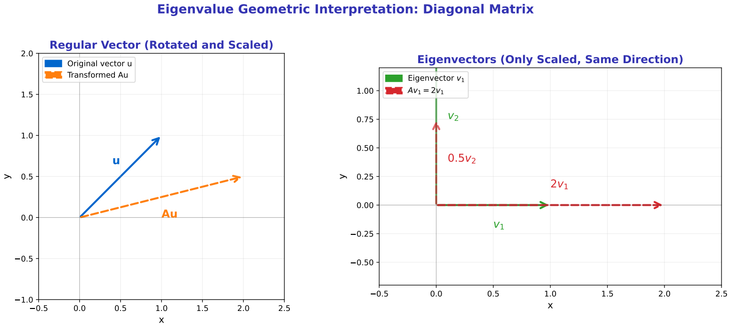
In words:

- A: a matrix (transformation)
- v: eigenvector (special direction)
- λ: eigenvalue (scaling factor)

Meaning: When matrix A acts on eigenvector v, the result is just a scaled version of v

No rotation – only stretching!

Eigenvectors are special directions that remain unchanged except for scaling



Visual: eigenvector direction is preserved under transformation

What is an Eigenvalue?

The eigenvalue λ

$$Av = \lambda v$$

Interpretation:

- $\lambda > 1$: vector is stretched
- $\lambda = 1$: vector unchanged
- $0 < \lambda < 1$: vector is shrunk
- $\lambda = 0$: vector collapses to zero
- $\lambda < 0$: vector flips direction and scales

Example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eigenvector: $(1, 0)$, Eigenvalue: $\lambda = 2$

The eigenvalue tells us how much the eigenvector is scaled

Why Are Eigenvalues Important?

Key applications:

1. **Understanding transformations:**
   - Eigenvectors show "principal directions"
   - Eigenvalues show "principal amounts of change"

2. **Long-term behavior:**
   - Powers of matrices: $A^n v = \lambda^n v$
   - Largest eigenvalue dominates over time

3. **Solving differential equations:**
   - $\frac{dx}{dt} = Ax$ has solutions involving eigenvalues

4. **Data analysis:**
   - Principal Component Analysis (PCA)
   - Finding patterns in high-dimensional data

Eigenvalues unlock the fundamental structure of linear transformations

Simple 2×2 Example: Step-by-Step

Example: Find eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Step 1: Set up the equation

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A - \lambda I)v = 0$$

Step 2: For nontrivial solutions We need $v \neq 0$, so the matrix $(A - \lambda I)$ must be singular:

$$\det(A - \lambda I) = 0$$

This equation determines the eigenvalues!

The characteristic equation gives us the eigenvalues

Simple 2×2 Example: Finding Eigenvalues

Step 3: Compute characteristic polynomial

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(2-\lambda) - 0 = (3-\lambda)(2-\lambda)$$

Step 4: Solve for eigenvalues

$$(3-\lambda)(2-\lambda) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 2$$

Result: Two eigenvalues found!

Upper triangular matrices have eigenvalues on the diagonal

Simple 2×2 Example: Finding Eigenvectors

Step 5: Find eigenvector for $\lambda_1 = 3$
Solve $(A - 3I)v = 0$:

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From row 1: $v_2 = 0$, v_1 is free
Eigenvector: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Step 6: Find eigenvector for $\lambda_2 = 2$
Solve $(A - 2I)v = 0$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From row 1: $v_1 = -v_2$
Eigenvector: $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or any scalar multiple)

Each eigenvalue has a corresponding eigenspace of eigenvectors

Verification: Check Our Work

Verify $\lambda_1 = 3$, $v_1 = (1, 0)$:

$$Av_1 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 v_1 \quad \checkmark$$

Verify $\lambda_2 = 2$, $v_2 = (-1, 1)$:

$$Av_2 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda_2 v_2 \quad \checkmark$$

Success! Both eigenvalue-eigenvector pairs are correct.

Always verify your eigenvalues and eigenvectors by checking $Av = \lambda v$

BSc Level: Computing Eigenvalues and Eigenvectors

Characteristic Polynomial

Definition
The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ is:

$$p(\lambda) = \det(A - \lambda I)$$

Properties:

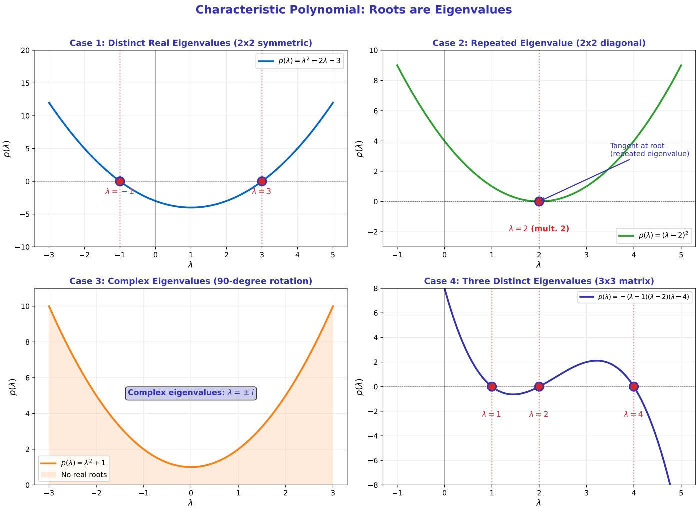
- $p(\lambda)$ is a polynomial of degree n
- Roots of $p(\lambda)$ are the eigenvalues of A
- Coefficients encode matrix properties

Key fact:

$$p(\lambda) = (-1)^n(\lambda^n - \text{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A))$$

where $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$ (trace)

The characteristic polynomial encodes all eigenvalue information



Roots of the characteristic polynomial are eigenvalues

Algorithm for finding eigenvalues:

Step 1: Form $A - \lambda I$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

Step 2: Compute $\det(A - \lambda I)$ Use cofactor expansion or special structure

Step 3: Solve $\det(A - \lambda I) = 0$ Find roots of the polynomial (may be complex!)

For $n \geq 5$, no closed-form solution exists in general (Abel-Ruffini theorem)

Finding Eigenvectors: General Procedure

After finding eigenvalue λ:

Step 1: Form the matrix $A - \lambda I$

Step 2: Solve the homogeneous system

$$(A - \lambda I)v = 0$$

Step 3: Find null space of $(A - \lambda I)$ Use Gaussian elimination to find basis vectors

Key points:

- The eigenspace $E_\lambda = \ker(A - \lambda I)$ is always non-empty (contains $v \neq 0$)
- Dimension of E_λ is the geometric multiplicity
- Any nonzero vector in E_λ is an eigenvector

The eigenspace is the null space of $A - \lambda I$

Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = -1$

Symmetric matrices always have real eigenvalues

Complete 2×2 Example: Eigenvectors

For $\lambda_1 = 3$:

$$A - 3I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Eigenvector: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -1$:

$$A + I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Eigenvector: $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Note: Eigenvectors are orthogonal! (property of symmetric matrices)

Symmetric matrices have orthogonal eigenvectors

Complete 3×3 Example

Find eigenvalues of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Characteristic polynomial (using cofactor expansion):

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (2-\lambda)[(2-\lambda)^2 - 1] - 1[1 \cdot (2-\lambda)] \\ &= (2-\lambda)^3 - 2(2-\lambda) - 1 \\ &= -(\lambda-2)^3 + 2(\lambda-2) - 1 \end{aligned}$$

After simplification: $-\lambda^3 + 6\lambda^2 - 10\lambda + 4 = 0$

3×3 systems require more computation but follow the same logic

Complete 3×3 Example: Results

Solving the cubic:

$$-\lambda^3 + 6\lambda^2 - 10\lambda + 4 = 0$$

We can verify by trial or numerical methods:

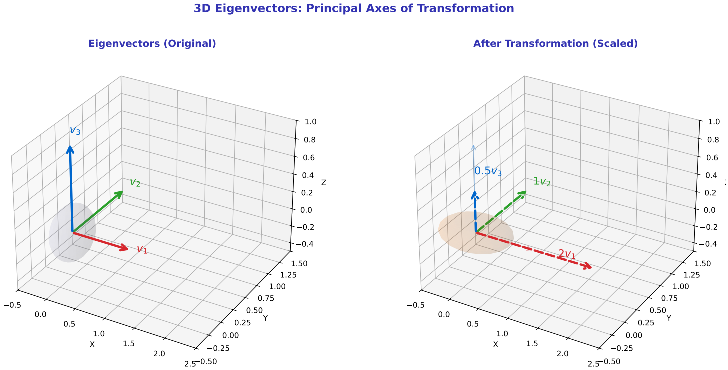
$$\lambda_1 = 2 + \sqrt{2}, \quad \lambda_2 = 2, \quad \lambda_3 = 2 - \sqrt{2}$$

Observations:

- Tridiagonal symmetric matrix (common in applications)
- All eigenvalues are real (symmetric property)
- Eigenvalues are well-separated

Finding eigenvectors: Same process as before, solve $(A - \lambda_i I)v = 0$ for each λ_i

Symmetric tridiagonal matrices appear frequently in numerical analysis



In 3D, eigenvectors define principal axes of transformation

Algebraic Multiplicity

Definition
The algebraic multiplicity of eigenvalue λ is the multiplicity of λ as a root of the characteristic polynomial.

Example 1:

$$p(\lambda) = (\lambda - 3)^2(\lambda + 1)$$

- $\lambda = 3$ has algebraic multiplicity 2
- $\lambda = -1$ has algebraic multiplicity 1

Example 2:

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic polynomial: $(5 - \lambda)^2(3 - \lambda)$
Algebraic multiplicities: $\lambda = 5$ (mult. 2), $\lambda = 3$ (mult. 1)

Algebraic multiplicity counts how many times an eigenvalue appears as a root

Definition
The geometric multiplicity of eigenvalue λ is the dimension of the eigenspace $E_\lambda = \ker(A - \lambda I)$.

Interpretation:

- Number of linearly independent eigenvectors for λ
- Dimension of the subspace of vectors unchanged in direction by A

Key theorem:

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

Example:

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

For $\lambda = 5$: $E_5 = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$
Geometric multiplicity $= 2$ (matches algebraic!)

Geometric multiplicity is the true dimension of the eigenspace

When Multiplicities Differ

Example where they differ:

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda I) = (5 - \lambda)^2 (3 - \lambda)$$

Algebraic multiplicity of $\lambda = 5$: 2

Eigenspace for $\lambda = 5$:

$$A - 5I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$E_5 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

Geometric multiplicity of $\lambda = 5$: 1

Consequence: Matrix is NOT diagonalizable!

When geometric multiplicity is less than algebraic, matrix is defective

Multiplicity Examples: Visual Comparison

Algebraic vs. Geometric Multiplicity: Comparison

Case 1: Multiplicities Match

Matrix A: diagonal(5, 5, 3)

Characteristic polynomial:
$p(\lambda) = (5-\lambda)^2(3-\lambda)$

Eigenvalue $\lambda = 5$:
Algebraic mult. = 2 | Geometric mult. = 2

Eigenspace:
Dim = 2 (spans first two coordinates)

STATUS: DIAGONALIZABLE

Multiplicities match - Full set of eigenvectors

Case 2: Multiplicities Differ

Matrix B: upper triangular
(5,1,0 on diag, 1 above diag)

Characteristic polynomial:
$p(\lambda) = (5-\lambda)^2(3-\lambda)$

Eigenvalue $\lambda = 5$:
Algebraic mult. = 2 | Geometric mult. = 1

Eigenspace:
Dim = 1 (only ONE independent eigenvector!)

STATUS: NOT DIAGONALIZABLE

DEFECTIVE MATRIX

Case 3: Mixed Multiplicities (Both Match)

Matrix C: diagonal(2, 3, 3)

$p(\lambda) = (2-\lambda)(3-\lambda)^2$

Eigenvalue Analysis:

λ	Alg. Mult.	Geom. Mult.
2	1	1
3	2	2

Both eigenvalues have matching multiplicities

Diagonalizable: YES

Full set of 3 independent eigenvectors

Case 4: Severely Defective Matrix

Matrix D: Jordan block
(4 on diag, 1 above diag)

$p(\lambda) = (4-\lambda)^3$

Single Eigenvalue $\lambda = 4$:
Algebraic mult. = 3 | Geometric mult. = 1

Eigenspace:
Dimension = 1
(Only 1 eigenvector for 3x3 matrix!)

SEVERELY DEFECTIVE
Needs generalized eigenvectors

Not Diagonalizable - Use Jordan Form

Comparing cases where multiplicities match vs. differ

Comprehensive Exercise 1

Problem: Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

Tasks:

1. Compute the characteristic polynomial
2. Find all eigenvalues
3. For each eigenvalue, find the eigenspace
4. Determine algebraic and geometric multiplicities
5. Verify $Av = \lambda v$ for each eigenvalue-eigenvector pair

Try this yourself before looking at the solution!

Practice is essential for mastering eigenvalue computation

Solution to Exercise 1

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix}$$

$$= (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

Step 2: Eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$

Step 3: Eigenvector for $\lambda_1 = 2$

$$A - 2I = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \to v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 4: Eigenvector for $\lambda_2 = 3$

$$A - 3I = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \to v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Multiplicities: Both algebraic and geometric multiplicities are 1 for each eigenvalue.

Always check dimensions of eigenspaces to determine geometric multiplicity

Comprehensive Exercise 2

Problem: Analyze the matrix

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Tasks:

1. Find the characteristic polynomial
2. What are the eigenvalues and their algebraic multiplicities?
3. Find all eigenvectors
4. What is the geometric multiplicity of $\lambda = 3$?
5. What does this tell us about diagonalizability?

Hint: This is a strictly upper triangular perturbation of $3I$

Upper triangular matrices have eigenvalues on the diagonal

Advanced Level: Eigenspaces and Spectral Theory

Definition
For eigenvalue λ, the eigenspace is:

$$E_\lambda = \ker(A - \lambda I) = \{v \in \mathbb{R}^n : Av = \lambda v\}$$

Properties:

- E_λ is a subspace (closed under addition and scalar multiplication)
- $\dim(E_\lambda) =$ geometric multiplicity of λ
- $E_\lambda \cap E_\mu = \{0\}$ for $\lambda \neq \mu$
- Eigenvectors from different eigenvalues are linearly independent

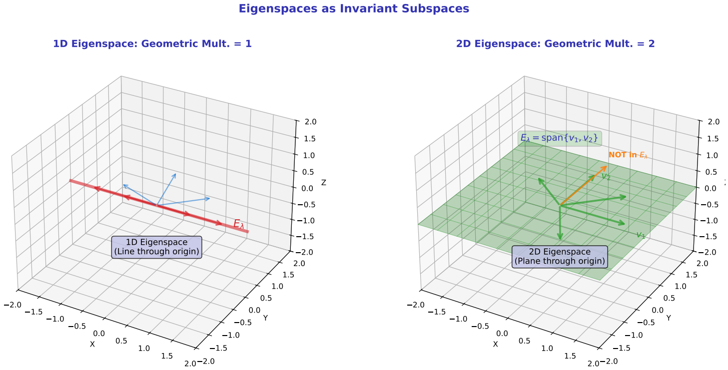
Decomposition: If A has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$:

$$\mathbb{R}^n = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

(if and only if A is diagonalizable)

Eigenspaces provide geometric structure for understanding linear transformations

Eigenspace Visualization



Eigenspaces as invariant subspaces under the transformation

Generalized Eigenvectors

Motivation: When geometric multiplicity is less than algebraic multiplicity, we need generalized eigenvectors.

Definition
Vector v is a generalized eigenvector of rank m for eigenvalue λ if:

$$(A - \lambda I)^m v = 0 \quad \text{but} \quad (A - \lambda I)^{m-1} v \neq 0$$

Example:

$$A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$$

Eigenvalue $\lambda = 5$ (algebraic mult. 2, geometric mult. 1)

Regular eigenvector: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Generalized eigenvector: Solve $(A - 5I)v_2 = v_1$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Generalized eigenvectors complete the basis when matrix is defective

Definition
A matrix A is defective if it does NOT have n linearly independent eigenvectors (for $n \times n$ matrix).

Equivalently:

- At least one eigenvalue has geometric multiplicity less than algebraic multiplicity
- Matrix is NOT diagonalizable

Example (defective):

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalue: $\lambda = 2$ (algebraic mult. 3)

Eigenspace: $E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ (geometric mult. 1)

Consequence: Use Jordan canonical form instead of diagonalization

Defective matrices require generalized eigenvectors for complete analysis

When do complex eigenvalues occur?

- Real matrices can have complex eigenvalues
- Always come in conjugate pairs: if $\lambda = a + bi$ is an eigenvalue, so is $\bar{\lambda} = a - bi$
- Common in rotation matrices and oscillatory systems

Example (rotation by 90 degrees):

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Eigenvectors (complex):

$$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Complex eigenvalues indicate rotational behavior

Interpreting Complex Eigenvalues

For complex eigenvalue $\lambda = a + bi$:

Real part a:

- $a > 0$: exponential growth (spiral outward)
- $a = 0$: pure oscillation (circle)
- $a < 0$: exponential decay (spiral inward)

Imaginary part b:

- Controls frequency of oscillation
- Larger $|b|$: faster rotation

Applications:

- Differential equations: $\frac{dx}{dt} = Ax$ has solutions $e^{at}(\cos(bt), \sin(bt))$
- Stability analysis: system stable if $\text{Re}(\lambda) < 0$ for all eigenvalues
- Quantum mechanics: complex eigenvalues in non-Hermitian systems

Complex eigenvalues encode oscillatory and rotational dynamics

Spectral Properties

The spectrum of A:

$$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

(set of all eigenvalues, counting multiplicities)

Key spectral properties:

1. Trace:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

2. Determinant:

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

3. Powers: If $Av = \lambda v$, then $A^k v = \lambda^k v$

4. Polynomials: If $p(\lambda) = c_0 + c_1\lambda + \cdots + c_k\lambda^k$, then:

$$p(A)v = p(\lambda)v$$

Spectral properties connect eigenvalues to global matrix characteristics

Advanced Exercise 1

Problem: Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Tasks:

1. Find the eigenvalues (they will be complex)
2. Find the complex eigenvectors
3. Interpret the geometric meaning (what transformation does A represent?)
4. Compute A^{10} using the eigenvalue decomposition

Hint: This matrix represents a rotation combined with scaling

Complex eigenvalues reveal rotational structure

Problem: Given

$$B = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Tasks:

1. Show that $\lambda = 4$ is the only eigenvalue
2. Determine algebraic and geometric multiplicities
3. Find all eigenvectors and generalized eigenvectors
4. Construct the Jordan canonical form
5. Explain why B is not diagonalizable

This is a classic defective matrix example

Jordan form is the generalization of diagonalization for defective matrices

Summary

Key concepts covered:

Beginner:

- Eigenvector equation: $Av = \lambda v$
- Geometric interpretation: stretching along special directions
- Simple computational examples

BSc:

- Characteristic polynomial: $\det(A - \lambda I) = 0$
- Finding eigenvalues and eigenvectors systematically
- Algebraic vs. geometric multiplicity

Advanced:

- Eigenspaces and generalized eigenvectors
- Complex eigenvalues and their interpretation
- Spectral properties and defective matrices

Eigenvalues reveal the fundamental structure of linear transformations

Beginner Level: Understanding Diagonalization

What is Diagonalization?

Definition
A matrix A is **diagonalizable** if it can be written as:

$$A = PDP^{-1}$$

where:

- D is a diagonal matrix (only entries on the main diagonal)
- P is an invertible matrix
- P^{-1} is the inverse of P

What does this mean?
We can express A in a simpler form where all the action happens on the diagonal.

Example:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Diagonalization transforms a matrix into its simplest form

Why Diagonalize? Power of Simplicity

Reason 1: Computing Powers
If $A = PDP^{-1}$, then:

$$A^n = PD^nP^{-1}$$

And computing D^n is trivial:

$$D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

Example:
Computing A^{100} directly: millions of operations
Computing via D^{100}: just two exponentiations!

Diagonalization makes matrix powers easy to compute

Why Diagonalize? Understanding Transformations

Reason 2: Understanding Transformations
$A = PDP^{-1}$ means:

1. P^{-1}: Change to eigenvector basis
2. D: Scale along eigendirections
3. P: Change back to standard basis

Reason 3: Solving Differential Equations
System $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ becomes simple:

$$\mathbf{x}(t) = Pe^{Dt}P^{-1}\mathbf{x}(0)$$

Reason 4: Understanding Long-term Behavior
Powers A^n grow/decay based on eigenvalues in D.

Diagonalization reveals the geometric and dynamic structure of transformations

When Can We Diagonalize?

The Key Question:
Not all matrices are diagonalizable!

Necessary Condition:
A must have n linearly independent eigenvectors (if A is $n \times n$).

Sufficient Conditions (guarantees diagonalizability):

- A has n distinct eigenvalues
- A is symmetric ($A = A^T$)

Examples:

- $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$: Already diagonal! Eigenvalues: 1, 2 (distinct)
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: Rotation by 90°, not diagonalizable over $\mathbb{R}$
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: Defective matrix, not diagonalizable

---

Diagonalizability depends on having enough eigenvectors

Simple 2×2 Example

Given:

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$$

Step 1: Find eigenvalues

Characteristic polynomial: $\det(A - \lambda I) = (4 - \lambda)(3 - \lambda) = 0$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 3$

Step 2: Find eigenvectors

For $\lambda_1 = 4$: $(A - 4I)\mathbf{v} = 0$ gives $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 3$: $(A - 3I)\mathbf{v} = 0$ gives $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Find eigenvalues first, then corresponding eigenvectors

Step 3: Construct P and D

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$

P has eigenvectors as columns, D has eigenvalues on diagonal.

Step 4: Compute P^{-1}

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Verification:

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} = A$$

Arrange eigenvectors as columns of P, eigenvalues on diagonal of D

BSc Level: Diagonalization Algorithm

Diagonalizability Condition

Theorem (Diagonalizability)
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Construction:
If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$, then:

$$A = PDP^{-1}$$

where:

$$P = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

The eigenvectors form the columns of P, eigenvalues the diagonal of D

Key Concepts:
Algebraic multiplicity of λ: How many times λ appears as a root of the characteristic polynomial.
Geometric multiplicity of λ: Dimension of eigenspace $E_\lambda = \ker(A - \lambda I)$.

Fundamental Inequality:

$$\text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

Diagonalizability Criterion:
A is diagonalizable if and only if for every eigenvalue λ:

$$\text{geometric multiplicity} = \text{algebraic multiplicity}$$

Consequence:
If all eigenvalues are distinct, A is automatically diagonalizable.

Matching multiplicities is the key test for diagonalizability

Diagonalization Algorithm

Input: $n \times n$ matrix A
Output: Matrices P, D such that $A = PDP^{-1}$ (if possible)

Algorithm:

1. Compute characteristic polynomial: $\det(A - \lambda I)$
2. Find all eigenvalues $\lambda_1, \ldots, \lambda_k$ (roots of char. polynomial)
3. For each eigenvalue λ_i:
   - Solve $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$
   - Find basis for eigenspace E_{λ_i}
4. Check: Do we have n linearly independent eigenvectors?
   - Yes: A is diagonalizable, proceed to step 5
   - No: A is not diagonalizable, STOP
5. Construct P (eigenvectors as columns) and D (eigenvalues on diagonal)
6. Compute P^{-1}

Systematic procedure to diagonalize any diagonalizable matrix

Complete 2×2 Example

Problem: Diagonalize $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 4)(\lambda - 2) = 0$$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$

Step 2: Eigenvector for $\lambda_1 = 4$

$$(A - 4I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$$

Solution: $v_1 = v_2$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Symmetric matrix is always diagonalizable with real eigenvalues

Complete 2×2 Example (Continued)

Step 3: Eigenvector for $\lambda_2 = 2$

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$$

Solution: $v_1 = -v_2$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Step 4: Construct P and D

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

Step 5: Compute P^{-1}
Using formula for 2×2 inverse:

$$P^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

Eigenvectors of symmetric matrix are orthogonal

Complete 3×3 Example

Problem: Diagonalize $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

Step 1: Characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)\det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)[(2-\lambda)(1-\lambda)] = (2-\lambda)^2(1-\lambda)$$

Eigenvalues: $\lambda_1 = 2$ (algebraic mult. 2), $\lambda_2 = 1$

Upper triangular matrix has eigenvalues on diagonal

Complete 3×3 Example: Eigenvectors

Step 2: Eigenvector for $\lambda_2 = 1$

$$(A - I)\mathbf{v} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

Solution: $v_1 = 0$, $v_2 + v_3 = 0$, so $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Step 3: Eigenspace for $\lambda_1 = 2$

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

Solution: $v_1 + v_3 = 0$, v_2 free

Eigenspace: $E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Geometric multiplicity $= 2 =$ Algebraic multiplicity!

We have 3 independent eigenvectors, so A is diagonalizable

Complete 3×3 Example: Diagonalization

Step 4: Construct P and D

Choose basis for E_2: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

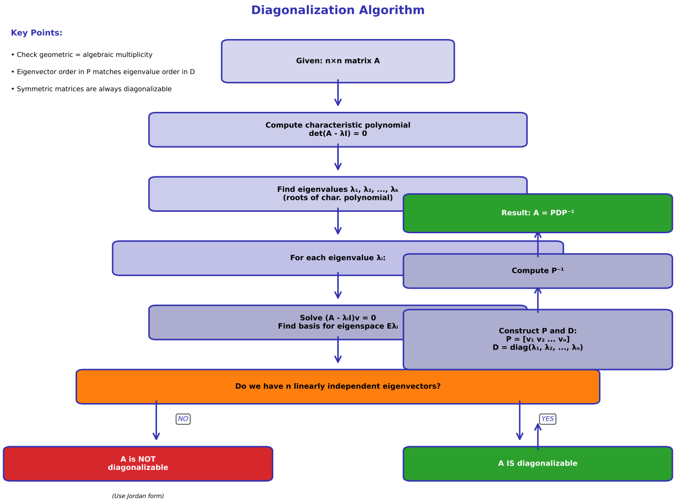
$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Note: Order of eigenvalues in D matches order of eigenvectors in P.

Verification: Compute P^{-1} and check $A = PDP^{-1}$.

Multiple eigenvectors for repeated eigenvalue allows diagonalization

Diagonalization Process: Step-by-Step



Flowchart showing complete diagonalization algorithm

Theorem (Matrix Powers)
If $A = PDP^{-1}$, then:

$$A^n = PD^nP^{-1}$$

Proof:

$$\begin{aligned}
A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1} \\
A^3 &= A^2 \cdot A = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1} \\
&\vdots \\
A^n &= PD^nP^{-1}
\end{aligned}$$

Key Insight:

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^n \end{pmatrix}$$

Diagonal matrices are trivial to exponentiate

Example: Computing A^{10}

Given: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ with diagonalization $A = PDP^{-1}$ where:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

Compute A^{10}:

$$D^{10} = \begin{pmatrix} 4^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} = \begin{pmatrix} 1048576 & 0 \\ 0 & 1024 \end{pmatrix}$$

$$\begin{aligned} A^{10} = PD^{10}P^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1048576 & 0 \\ 0 & 1024 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} 524800 & 523776 \\ 523776 & 524800 \end{pmatrix} \end{aligned}$$

Direct computation would require millions of operations!

Matrix Powers: Direct vs Diagonalization Comparison

Method 1: Direct Computation

Compute A^n directly:

$A^2 = A \times A$
$A^3 = A^2 \times A$
$A^4 = A^3 \times A$
...
$A^n = A^{(n-1)} \times A$

Operations:
- n-1 matrix multiplications
- Each multiplication: $O(d^3)$
- Total: $O(n \cdot d^3)$

For A^100:
- 99 matrix multiplications
- For 2×2: ~800 operations
- For 100×100: ~10^8 operations!

Method 2: Diagonalization

Compute via diagonalization:

Step 1: Diagonalize $A = PDP^{-1}$
(one-time cost: $O(d^3)$)

Step 2: Compute D^n
$D^n = \text{diag}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_d^n)$
(d exponentiations: $O(d)$)

Step 3: $A^n = PD^nP^{-1}$
(2 matrix mult.: $O(d^3)$)

Total: $O(d^3 + d)$

For A^100:
- Diagonalize once
- 2 exponentiations
- 2 matrix multiplications
- For 100×100: ~2×10^6 operations

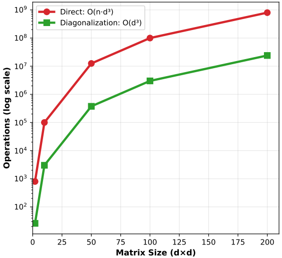
Complexity Comparison

Matrix Size	Direct A^100	Diagonalization
2×2	800	50
10×10	10^5	2×10^3
100×100	10^8	2×10^6
1000×1000	10^{11}	2×10^9

Speedup for 100×100 matrix:

~50× faster!

Computational Cost for A^100



Example: Computing A^10

Given A = [[3, 1], [1, 3]]

Diagonalization:

$\lambda_1 = 4.0$, $\lambda_2 = 2.0$
P = [[0.71, -0.71]
[0.71, 0.71]]

Compute D^10:

D^10 = [[1048576, 0]
[0, 1024]]

Result $A^{10} = PD^{10}P^{-1}$:

[[524800, 523776]
[523776, 524800]]

When to Use Each Method?

Use DIRECT method when:
- Small n (n < 5)
- One-time computation
- Matrix not diagonalizable
- Very small matrices (2×2)

Use DIAGONALIZATION when:
- Large n (n ≥ 10)
- Multiple powers needed
- Large matrices (d > 10)
- Matrix is diagonalizable

Key Insight:
Diagonalization has one-time setup cost, but then computing any power A^n is very fast!

Computational complexity: Direct $O(n^4)$ vs Diagonalization $O(n^3 + n)$

Example of Non-Diagonalizable Matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Analysis:
Characteristic polynomial: $\det(A - \lambda I) = (1 - \lambda)^2 = 0$
Eigenvalue: $\lambda = 1$ (algebraic multiplicity 2)

Eigenspace:

$$(A - I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$$

Solution: $v_2 = 0$, so $E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
Geometric multiplicity $= 1 < 2 =$ Algebraic multiplicity

Conclusion: Not diagonalizable! (Called a **defective matrix**)

Defective matrices lack enough eigenvectors for diagonalization

Warning Signs:

1. Repeated eigenvalue with small eigenspace
2. Geometric multiplicity $<$ Algebraic multiplicity
3. Matrix A such that $(A - \lambda I)^k \neq 0$ but $(A - \lambda I) = 0$ has small rank

Common Examples:

- Strict upper/lower triangular matrices (except identity)
- Jordan blocks: $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$
- Some rotation matrices (over $\mathbb{R}$)

What to do?
Use Jordan Normal Form (covered in advanced section).

Not all matrices are diagonalizable, but all have a Jordan form

Exercises: Diagonalization

Exercise 1
Diagonalize the following matrix:

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

Find P, D, and P^{-1}.

Exercise 2
Is the following matrix diagonalizable?

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

If yes, find the diagonalization. If no, explain why.

Practice computing eigenvalues, eigenvectors, and checking diagonalizability

Exercise 3

Given $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 3$:

- Find the diagonalization $A = PDP^{-1}$
- Compute A^{10} using diagonalization

Exercise 4

For the matrix $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

- Show that C is not diagonalizable over $\mathbb{R}$
- What happens if we allow complex eigenvalues?

Explore computational techniques and limitations

Advanced Level: Spectral Theory

Theorem (Spectral Decomposition)
If A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then:

$$A = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T$$

(assuming eigenvectors are normalized: $\|\mathbf{v}_i\| = 1$)

Interpretation:
A is a weighted sum of rank-1 projections onto eigenspaces.

Each term $\mathbf{v}_i\mathbf{v}_i^T$:

- Is a projection matrix onto span$\{\mathbf{v}_i\}$
- Has rank 1
- Satisfies $(\mathbf{v}_i\mathbf{v}_i^T)^2 = \mathbf{v}_i\mathbf{v}_i^T$ (idempotent)

Spectral decomposition expresses A as sum of rank-1 projections

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$
Normalized eigenvectors: $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

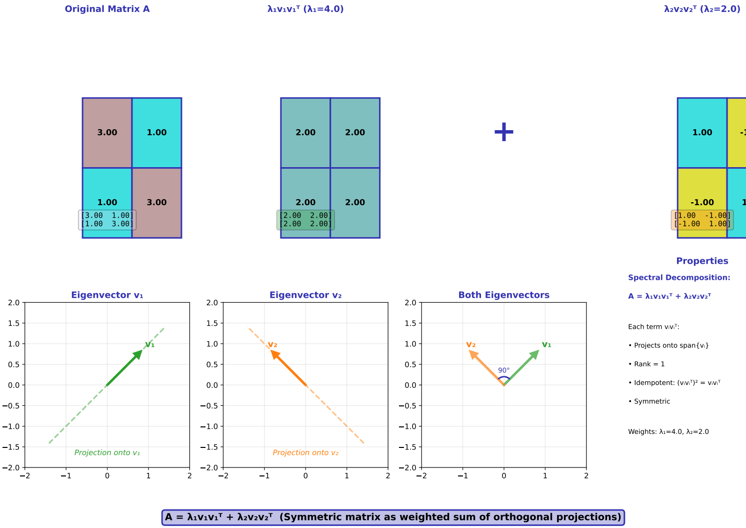
Spectral decomposition:

$$A = 4 \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2 \cdot \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Each eigenvalue contributes a scaled projection onto its eigenspace

Spectral Decomposition: Visualization



Matrix as weighted sum of rank-1 projections

Definition (Similarity)
Matrices A and B are **similar** if there exists an invertible P such that:

$$B = P^{-1}AP$$

Diagonalization means: A is similar to diagonal matrix D.

Similarity Invariants (properties preserved under similarity):

1. Determinant: $\det(B) = \det(A)$
2. Trace: $\text{tr}(B) = \text{tr}(A)$
3. Characteristic polynomial: $\det(B - \lambda I) = \det(A - \lambda I)$
4. Eigenvalues (with multiplicities)
5. Rank: $\text{rank}(B) = \text{rank}(A)$

Similar matrices represent the same transformation in different bases

Geometric Interpretation:
If A and B are similar via $B = P^{-1}AP$, then:

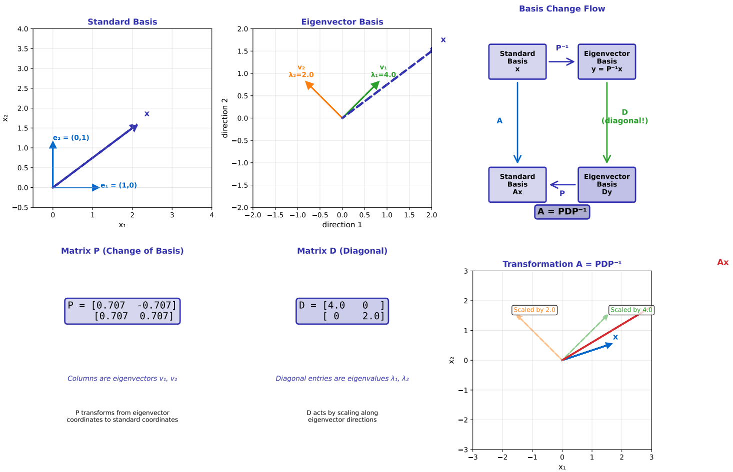
- A represents a linear transformation T in basis $\mathcal{B}_1$
- B represents the same transformation T in basis $\mathcal{B}_2$
- P is the change-of-basis matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$

Diagonalization as Optimal Basis:
Choosing $\mathcal{B}_2$ to be the eigenvector basis makes $B = D$ diagonal!

Consequence:
All diagonalizable $n \times n$ matrices with the same eigenvalues (counting multiplicities) are similar to each other.

Similarity is basis change; diagonalization finds the best basis

Basis Change Flow: Geometric View



Similarity transformation via basis change from standard to eigenvector basis

Problem: Solve the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

Solution via Diagonalization:
If $A = PDP^{-1}$, change variables: $\mathbf{y} = P^{-1}\mathbf{x}$
Then:

$$\frac{d\mathbf{y}}{dt} = P^{-1}\frac{d\mathbf{x}}{dt} = P^{-1}A\mathbf{x} = P^{-1}(PDP^{-1})\mathbf{x} = D\mathbf{y}$$

This is a **decoupled** system:

$$\frac{dy_i}{dt} = \lambda_i y_i \quad \Rightarrow \quad y_i(t) = e^{\lambda_i t} y_i(0)$$

General solution:

$$\mathbf{x}(t) = P\mathbf{y}(t) = Pe^{Dt}P^{-1}\mathbf{x}_0$$

where $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$

Diagonalization decouples systems of ODEs

Example: 2D Dynamical System

Problem:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Diagonalization: $A = PDP^{-1}$ with $\lambda_1 = 4$, $\lambda_2 = 2$

Solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} \end{pmatrix}$$

Long-term behavior: Dominated by e^{4t} term (largest eigenvalue).

Eigenvalues determine stability and growth rates

Stability Analysis via Eigenvalues

System: $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$
Stability Classification:

- **Stable node:** All $\text{Re}(\lambda_i) < 0 \Rightarrow \mathbf{x}(t) \to \mathbf{0}$
- **Unstable node:** Some $\text{Re}(\lambda_i) > 0 \Rightarrow \|\mathbf{x}(t)\| \to \infty$
- **Saddle point:** Mixed signs of $\text{Re}(\lambda_i)$
- **Center:** Pure imaginary eigenvalues $\Rightarrow$ oscillations

Example Applications:

- Population dynamics (predator-prey models)
- Mechanical systems (coupled oscillators)
- Electrical circuits (RLC networks)
- Chemical reaction kinetics

Diagonalization provides complete solution and stability analysis

What if A is not diagonalizable?
Every matrix A is similar to a **Jordan Normal Form**:

$$A = PJP^{-1}$$

where J is block-diagonal:

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$$

Jordan Block:

$$J_i = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

Jordan form is the closest thing to diagonal for non-diagonalizable matrices

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
Not diagonalizable, but has Jordan form:

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(Already in Jordan form!)

Matrix Powers:
For Jordan block $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$:

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

General Pattern:
Jordan form allows computing matrix functions $f(A)$ even when A is defective.

Jordan theory extends diagonalization to all matrices

Advanced Exercises

Exercise 1: Spectral Decomposition

For $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$:

- Find the spectral decomposition $A = \sum \lambda_i \mathbf{v}_i \mathbf{v}_i^T$
- Verify that each $\mathbf{v}_i \mathbf{v}_i^T$ is a projection matrix

Exercise 2: Differential Equations

Solve $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ using diagonalization.
Determine the stability of the system.

Apply spectral theory to concrete problems

Exercise 3: Similarity Invariants
Prove that if A and B are similar, then:

- $\mathrm{tr}(A) = \mathrm{tr}(B)$
- $\det(A) = \det(B)$

Exercise 4: Jordan Form
Find the Jordan Normal Form of:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Compute A^{10} using the Jordan form.

Explore theoretical foundations and computational techniques

Summary: Diagonalization

Key Concepts:

1. **Diagonalizability:** $A = PDP^{-1}$ requires n independent eigenvectors
2. **Algorithm:** Find eigenvalues $\rightarrow$ Find eigenvectors $\rightarrow$ Check independence $\rightarrow$ Construct P and D
3. **Matrix Powers:** $A^n = PD^nP^{-1}$ (diagonal powers are trivial)
4. **Spectral Decomposition:** $A = \sum \lambda_i \mathbf{v}_i \mathbf{v}_i^T$
5. **Applications:** Differential equations, stability analysis, matrix functions

Diagonalizability Conditions:

- Sufficient: n distinct eigenvalues OR symmetric matrix
- Necessary: Geometric multiplicity = Algebraic multiplicity for all eigenvalues

Diagonalization transforms matrices to their simplest form in the eigenvector basis

When to Use Diagonalization:

- Computing high powers A^n
- Solving systems of differential equations
- Understanding long-term behavior
- Matrix exponential e^{At}
- Principal Component Analysis (PCA)

Limitations:

- Not all matrices are diagonalizable
- Complex eigenvalues (for real matrices)
- Numerical stability issues for nearly defective matrices

Next Steps:

- Symmetric matrices: Always diagonalizable, orthogonal eigenvectors
- Singular Value Decomposition (SVD): Works for all matrices
- Jordan Normal Form: For non-diagonalizable matrices

Diagonalization is fundamental to spectral theory and numerical linear algebra

BSc Level: Markov Chains

Stochastic Matrices: Definition

What is a stochastic matrix?
A matrix $P \in \mathbb{R}^{n\times n}$ is called **stochastic** (or transition matrix) if:

1. All entries are non-negative: $p_{ij} \geq 0$
2. Each **column** sums to 1:

$$\sum_{i=1}^{n} p_{ij} = 1 \quad \text{for all } j$$

Interpretation:

- p_{ij} = probability of moving from state j to state i
- Column j represents all possible transitions FROM state j
- Probabilities must sum to 1 (something must happen!)

Example:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

Stochastic matrices encode transition probabilities between states

Markov Chains: States and Transitions

Definition

A **Markov chain** is a stochastic process where:

- System has finitely many **states**
- Transitions between states are probabilistic
- **Markov property**: Future depends only on present, not on past

Mathematical model:

$$x_{k+1} = Px_k$$

where:

- $x_k \in \mathbb{R}^n$: probability distribution at time k
- P: stochastic transition matrix
- $(x_k)_i$ = probability of being in state i at time k

Key property: If x_0 is a probability distribution (entries sum to 1), then so is x_k for all k!

Markov chains model memoryless random processes

Example: Weather Model

Simple weather model with 2 states:

- State 1: Sunny
- State 2: Rainy

Transition rules:

- If sunny today: 70% chance sunny tomorrow, 30% chance rainy
- If rainy today: 40% chance sunny tomorrow, 60% chance rainy

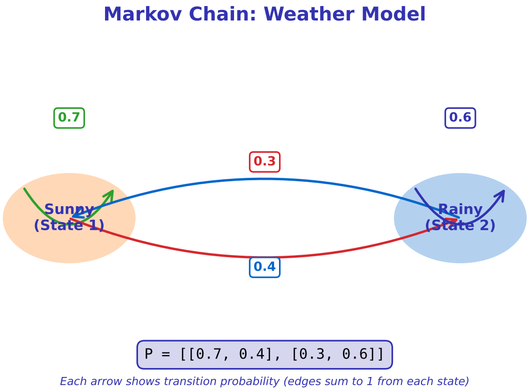
Transition matrix:

$$P = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$$

Column 1 (from sunny): $[0.7, 0.3]^T$ (probabilities to sunny/rainy) Column 2 (from rainy): $[0.4, 0.6]^T$

Question: If it's sunny today ($x_0 = [1, 0]^T$), what's the probability distribution tomorrow?

Weather transitions can be modeled as a Markov chain



State transition diagram shows probabilities as directed edges

Steady-State Vector: The Long-Term Behavior

Key question:
What happens to x_k as $k \to \infty$? Does it converge?

Definition: Steady-state vector
A vector π is called a **steady-state** (or stationary distribution) if:

$$P\pi = \pi$$

This means: π is an **eigenvector of P with eigenvalue** $\lambda = 1$!

Interpretation:

- Once the system reaches state π, it stays in π (in distribution)
- Long-term equilibrium
- Independent of initial state x_0 (under mild conditions)

Theorem: Every stochastic matrix has $\lambda = 1$ as an eigenvalue

---

Steady state is the dominant eigenvector with eigenvalue 1

Method 1: Solve eigenvalue problem
Find eigenvector for $\lambda = 1$:

$$(P - I)\pi = 0$$

Normalize so that $\sum_i \pi_i = 1$

Method 2: Power method
Start with any initial distribution x_0 and iterate:

$$x_{k+1} = Px_k$$

Under mild conditions: $x_k \to \pi$ as $k \to \infty$

Convergence condition:
For convergence to unique steady state, P must be:

- **Irreducible**: Can reach any state from any other state
- **Aperiodic**: No deterministic cycles

Two equivalent methods: eigenvalue problem or power iteration

Worked Example: Weather Model Steady State

Given:

$$P = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$$

Method 1: Solve $(P - I)\pi = 0$

$$\begin{pmatrix} -0.3 & 0.4 \\ 0.3 & -0.4 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation: $-0.3\pi_1 + 0.4\pi_2 = 0 \Rightarrow \pi_1 = \frac{4}{3}\pi_2$
With constraint $\pi_1 + \pi_2 = 1$:

$$\frac{4}{3}\pi_2 + \pi_2 = 1 \Rightarrow \pi_2 = \frac{3}{7}, \quad \pi_1 = \frac{4}{7}$$

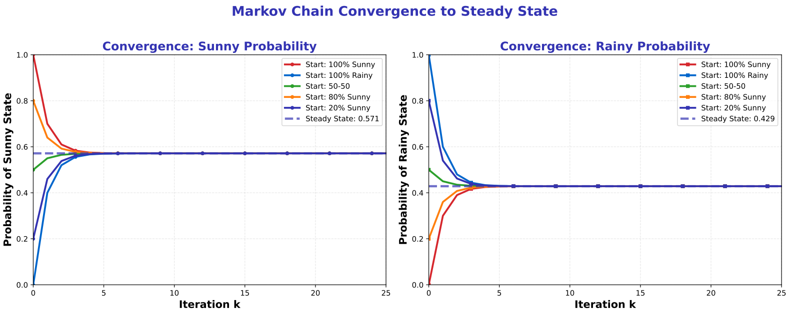
Result:

$$\pi = \begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix} \approx \begin{pmatrix} 0.571 \\ 0.429 \end{pmatrix}$$

Long-term: 57.1% sunny days, 42.9% rainy days!

Steady state gives long-term probability distribution

Convergence to Steady State



Probability vectors converge to steady state regardless of initial condition

Problem:
Two competing brands A and B. Each month:

- 80% of brand A customers stay with A, 20% switch to B
- 60% of brand B customers stay with B, 40% switch to A

Tasks:

1. Write the transition matrix P
2. Verify it is stochastic (columns sum to 1)
3. Find the steady-state distribution
4. If market share is currently 50-50, what will it be in the long run?

Hint: Set up states as: State 1 $=$ Brand A customer, State 2 $=$ Brand B customer

Solution on next slide

Market share models are classic applications of Markov chains

Solution: Brand Switching Model

Step 1: Transition matrix

$$P = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$$

Column 1 (from A): $[0.8, 0.2]^T$ (stay A / switch to B) Column 2 (from B): $[0.4, 0.6]^T$ (switch to A / stay B)

Step 2: Verify stochastic
$0.8 + 0.2 = 1$ (Y), $0.4 + 0.6 = 1$ (Y)

Step 3: Solve $(P - I)\pi = 0$

$$-0.2\pi_1 + 0.4\pi_2 = 0 \Rightarrow \pi_1 = 2\pi_2$$

With $\pi_1 + \pi_2 = 1$: $\pi = [2/3, 1/3]^T = [0.667, 0.333]^T$

Answer: Long term: Brand A will have 66.7% market share, Brand B 33.3%

Brand A's higher retention rate leads to dominance in steady state

Advanced Level: PageRank Algorithm

PageRank: The Web as a Graph

The problem: How to rank web pages?
In 1998, Google founders Larry Page and Sergey Brin asked:
"Which web pages are most important?"

Key insight:
A page is important if important pages link to it!

Mathematical model:

- Nodes = web pages
- Directed edges = hyperlinks
- Importance = steady-state distribution of a random walk

Connection to linear algebra:
PageRank is the **dominant eigenvector** of the web's link matrix!

PageRank revolutionized web search by using graph structure

PageRank Intuition: Links as Votes

The random surfer model:
Imagine a user randomly clicking links:

1. Start at any page
2. With probability 0.85: click a random outgoing link
3. With probability 0.15: jump to any random page ("teleport")
4. Repeat forever

PageRank = Fraction of time spent on each page

Why this makes sense:

- Links from important pages count more
- Pages with many incoming links rank higher
- Prevents manipulation (teleportation breaks cycles)

Key principle:

$$\text{PageRank}(p) = \sum_{q \to p} \frac{\text{PageRank}(q)}{\text{OutDegree}(q)}$$

Each page distributes its PageRank equally among its outgoing links

Random surfer model makes PageRank recursive and well-defined

PageRank: Mathematical Formulation

Link matrix construction:
For n pages, define matrix L:

$$L_{ij} = \begin{cases} 1/\text{outdeg}(j) & \text{if page } j \text{ links to page } i \\ 0 & \text{otherwise} \end{cases}$$

Column j represents how page j distributes its PageRank

Problem: L might not be stochastic (dangling nodes)

Google matrix with damping:

$$G = \alpha L + (1-\alpha)\frac{1}{n}\mathbf{1}\mathbf{1}^T$$

where:

- $\alpha = 0.85 =$ damping factor (follow links)
- $1 - \alpha = 0.15 =$ teleportation probability
- $\mathbf{1}\mathbf{1}^T/n =$ uniform distribution matrix

PageRank vector: Steady state of G, i.e., solve $G\pi = \pi$

Google matrix ensures unique, positive steady state

Power Method for PageRank

Algorithm:

1. Initialize: $\pi^{(0)} = \mathbf{1}/n$ (uniform distribution)
2. Iterate until convergence:

$$\pi^{(k+1)} = G\pi^{(k)}$$

3. Stop when $\|\pi^{(k+1)} - \pi^{(k)}\| < \epsilon$

Why it works:

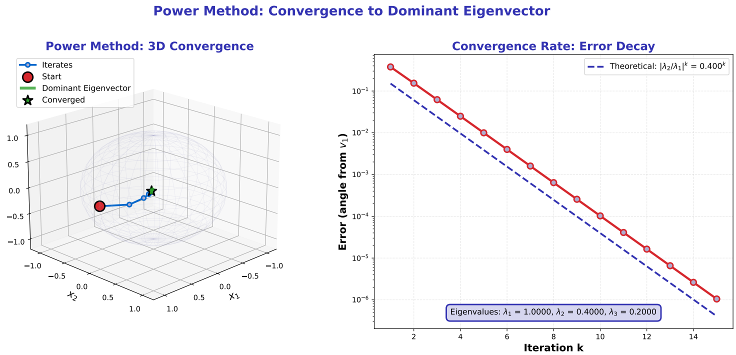
- G is stochastic, irreducible, aperiodic
- Eigenvalue $\lambda = 1$ is simple and dominant
- Power method converges to dominant eigenvector

Computational complexity:

- Each iteration: $O(n \cdot \text{links})$ (sparse matrix-vector product)
- Convergence: typically 50-100 iterations
- For web-scale ($n \approx 10^{10}$): needs distributed computing

Power method scales to billions of web pages

Power Method: Convergence to Dominant Eigenvector



Iterates converge exponentially to the direction of dominant eigenvector

Network structure:

- Page 1 links to: 2, 3
- Page 2 links to: 1, 3
- Page 3 links to: 1
- Page 4 links to: 1, 2, 3 (hub page)

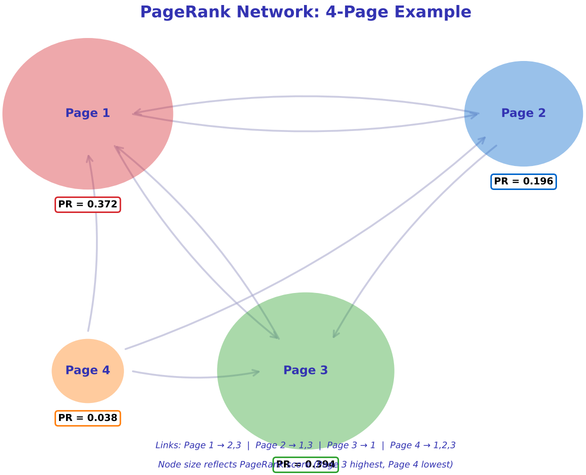
Link matrix L:

$$L = \begin{pmatrix} 0 & 1/2 & 1 & 1/3 \\ 1/2 & 0 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: Page 4 has no outgoing links (column 4 all zeros)!
This is a **dangling node** - needs special handling

Small example illustrates PageRank computation

PageRank Network: Link Structure Visualization



Node sizes reflect PageRank scores after convergence

Handling Dangling Nodes and Teleportation

Problem 1: Dangling nodes
Pages with no outgoing links cause columns of zeros in L
Solution: Replace zero columns with $\mathbf{1}/n$ (uniform teleport)

Problem 2: Spider traps
Disconnected components can trap random surfer
Solution: Add teleportation to ALL pages

Complete Google matrix:

$$G = \alpha \left(L + \frac{1}{n}\mathbf{d}\mathbf{1}^T \right) + (1-\alpha)\frac{1}{n}\mathbf{1}\mathbf{1}^T$$

where $\mathbf{d}_j = 1$ if page j is dangling, 0 otherwise

Properties of G:

- Stochastic (all columns sum to 1)
- Positive entries (teleportation ensures connectivity)
- Unique dominant eigenvector

Teleportation ensures well-defined PageRank for any graph

Computing PageRank: 4-Page Example

With damping $\alpha = 0.85$:
Fix dangling node and add teleportation:

$$G = 0.85 \cdot \begin{pmatrix} 0 & 1/2 & 1 & 1/4 \\ 1/2 & 0 & 0 & 1/4 \\ 1/2 & 1/2 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} + 0.15 \cdot \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

Power method iterations:
Start with $\pi^{(0)} = [0.25, 0.25, 0.25, 0.25]^T$
After 20 iterations:

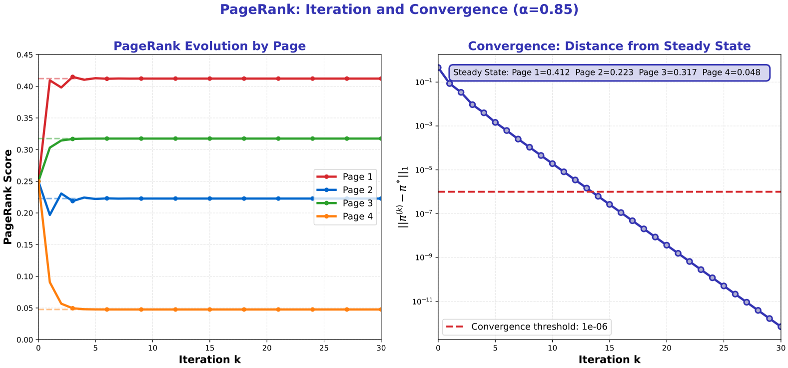
$$\pi \approx [0.372, 0.196, 0.394, 0.038]^T$$

Interpretation:

- Page 3 most important (39.4%) - receives links from all others
- Page 1 second (37.2%) - hub and target
- Page 4 least important (3.8%) - no outgoing links

PageRank quantifies page importance based on link structure

PageRank Iteration: Convergence Over Time



PageRank scores converge exponentially to steady state

Exercise: Compute PageRank for Small Network

Network:

- Page A links to: B, C
- Page B links to: C
- Page C links to: A

Tasks:

1. Construct the link matrix L
2. Verify column sums equal 1
3. Build Google matrix G with $\alpha = 0.85$
4. Run 10 iterations of power method starting from $\pi^{(0)} = [1/3, 1/3, 1/3]^T$
5. Which page has highest PageRank?

Hints:

- Draw the network diagram first
- Page A splits its PageRank equally between B and C
- Use a calculator or write simple Python code

Practice computing PageRank by hand for small examples

Summary: From Markov Chains to PageRank

Key Concepts:

1. **Stochastic Matrices**
   - Non-negative entries, columns sum to 1
   - Model transition probabilities
2. **Markov Chains**
   - $x_{k+1} = Px_k$: memoryless random walks
   - Steady state = eigenvector with $\lambda = 1$
   - Long-term behavior independent of initial state
3. **PageRank**
   - Web as directed graph
   - PageRank = steady state of random surfer
   - Damping factor prevents manipulation
   - Power method computes dominant eigenvector

Eigenvalues connect probability theory, dynamical systems, and web search

Applications and Extensions

Applications of Markov Chains:

- Population dynamics and genetics
- Queuing theory and operations research
- Natural language processing (n-gram models)
- Financial modeling (credit ratings)
- Physics (statistical mechanics)

Extensions of PageRank:

- **Personalized PageRank**: Different teleportation preferences
- **Topic-sensitive PageRank**: Bias toward specific topics
- **Social network analysis**: Influence and centrality measures
- **Recommendation systems**: Item-item similarity
- **Biological networks**: Protein importance

Modern developments:
Neural network-based ranking (BERT, neural PageRank) augment but don't replace the core eigenvector idea

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The eigenvector perspective unifies many ranking and importance measures

Further Topics: Advanced Theory

Convergence rate:
Power method converges at rate $|\lambda_2/\lambda_1|$ where:

- $\lambda_1 = 1$ (dominant eigenvalue)
- $\lambda_2 =$ second-largest eigenvalue

For Google matrix: $|\lambda_2| \leq \alpha = 0.85$

Perron-Frobenius Theorem:
For positive stochastic matrix P:

- $\lambda = 1$ is simple eigenvalue
- Corresponding eigenvector has positive entries
- All other eigenvalues satisfy $|\lambda| < 1$

Sparse matrix techniques:

- Web matrix has ≈ 10 nonzeros per row (billions of pages)
- Exploit sparsity: never form dense G explicitly
- Store only L, compute teleportation on-the-fly

Advanced theory ensures PageRank is well-defined and computable

Conclusion: The Power of Eigenvalues

What we learned:

- Eigenvalues and eigenvectors describe **long-term behavior**
- Dominant eigenvector = steady state = equilibrium
- Power method is simple yet powerful
- Applications span probability, physics, and computer science

Big picture:

Eigenvalues transform qualitative questions
("Which page is most important?")
into quantitative linear algebra
("Find the dominant eigenvector")

This is the essence of mathematical modeling!

Linear algebra provides the language for understanding complex systems

Summary: Eigenvalues and Applications

Foundations

- Complex numbers extend real numbers to handle all polynomial roots
- Similar matrices share eigenvalues and spectral properties

Eigenvalue Theory

- Eigenvectors: directions preserved under transformation ($\mathrm{Av} = \lambda \mathrm{v}$)
- Characteristic polynomial $\det(A - \lambda I) = 0$ yields eigenvalues
- Algebraic multiplicity (polynomial) vs geometric multiplicity (eigenspace dimension)

Diagonalization

- $A = PDP^{-1}$ when A has n independent eigenvectors
- Enables efficient computation of A^n, stability analysis, differential equations
- Spectral decomposition: $A = \sum \lambda_i v_i v_i^T$

Applications

- Markov chains: steady-state as eigenvector for $\lambda = 1$
- PageRank: web importance via dominant eigenvector of stochastic matrix

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Eigenvalues are central to modern applied mathematics and data science