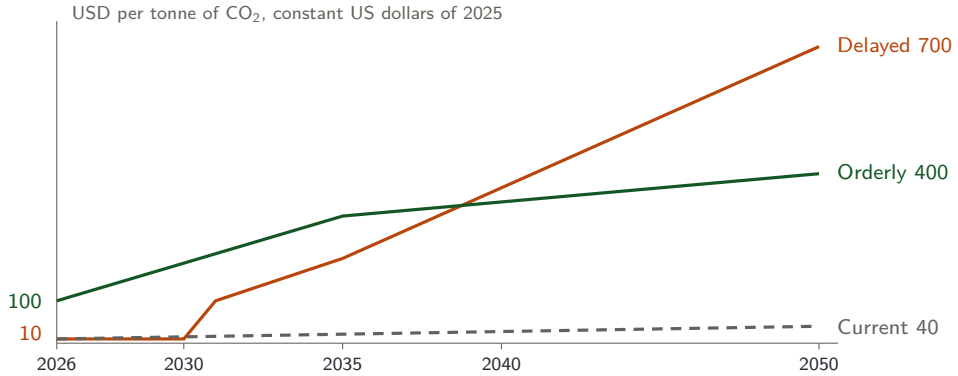
Carbon price paths and the present value of carbon costs

Mathematics of green finance risk management

Joerg Osterrieder

University of Twente

Three carbon price paths



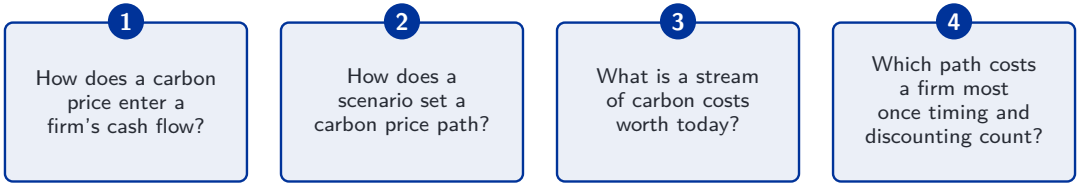
Rank the three carbon price paths by what they cost a firm per Mt of CO_2 it emits each year from 2026 to 2050, counted today at a real rate of 5 per cent.

Stylised values.

Abatement, scenario paths and discount factors set the present value of carbon costs

1. Compute a firm's abatement, compliance cost and net carbon cost at a carbon price
2. Compute a scenario price between two of its points, and a Hotelling price
3. Convert a nominal rate and inflation into the real rate for constant-dollar costs
4. Compute discount factors and the present value of a yearly carbon cost
5. Derive the growing annuity from a geometric sum and apply it to a cost
6. Rank carbon price paths by present value, and explain how the discount rate and abatement change the ranking

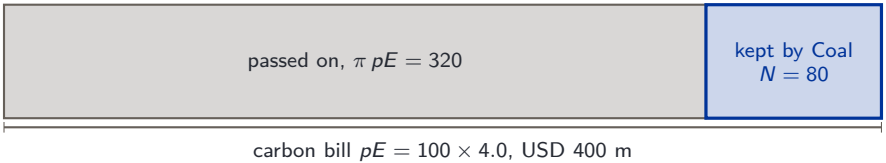
Four questions on carbon costs



A carbon price p sets a firm's abatement, its compliance cost C and the net cost N it keeps. A scenario sets p for every year from 2026 to 2050. Discount factors turn each year's N into a present value, and the present values rank the three paths.

A carbon price times emissions is a cost the firm passes on in part

1 2 3 4 **How does a carbon price enter a firm's cash flow?**



The six-firm book: six stylised firms, Coal, Cement, Steel, Port, Rice and Solar, each with its own inputs.

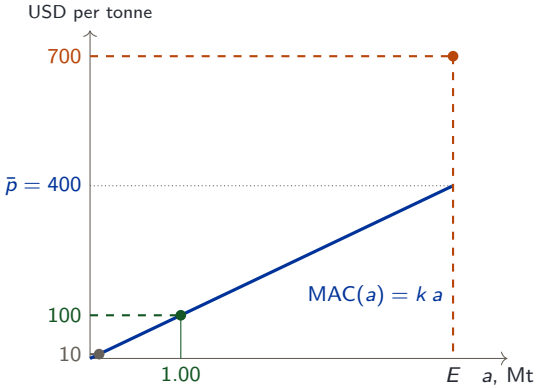
Coal, a coal power plant, emits $E = 4.0$ Mt a year; at a carbon price p of USD 100 per tonne its carbon bill is pE. It passes a share $\pi = 0.80$ on to customers and keeps the net cost

$$N = (1 - \pi)\, pE = 0.20 \times 100 \times 4.0 = 80.$$

Stylised values.

What does Coal's own cash flow lose at the Current path's 2050 price of USD 40, if it abates nothing?

A firm abates until its marginal abatement cost reaches the carbon price



Abating a Mt costs more the more the firm has already abated: the marginal abatement cost is $\text{MAC}(a) = k\,a$. Coal abates all its $E = 4.0$ Mt at the full-abatement price $\bar{p} = 400$, so $k = \bar{p}/E = 100$.

The firm abates while the next Mt costs less than the carbon price:

$$a^* = \min(p/k,\ E).$$

At USD 10: 0.10 Mt. At 100: 1.00 Mt. At 700, $p/k = 7$ Mt exceeds E, and Coal abates 4.0 Mt.

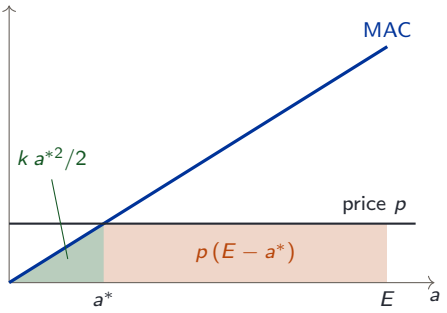
Stylised values.

Cement emits 3.0 Mt and abates fully at USD 1,200. How many Mt does it abate at the Delayed 2050 price?

Compliance cost adds the carbon bill on unabated tonnes to the abatement triangle

$$
\begin{aligned}
C(p) &= p\,(E - a^*) + \tfrac{1}{2}\, a^* \cdot k\, a^* && \text{rectangle plus triangle area} \\
&= p\,(E - a^*) + \frac{k\, a^{*2}}{2} && \text{multiply out} \\
&= pE - \frac{p^2}{k} + \frac{k}{2}\,\frac{p^2}{k^2} && \text{substitute } a^* = p/k \\
&= pE - \frac{p^2}{2k} && \text{collect the } p^2 \text{ terms} \\
N(p) &= (1 - \pi)\, C(p) && \text{net cost}
\end{aligned}
$$

The compliance cost C is the price on the tonnes the firm still emits plus the abatement cost of the tonnes it cuts. The third line holds while $a^* < E$, that is while $p < \bar{p}$.



Orange: the carbon bill on unabated tonnes. Green: the cost of the abated tonnes, the area under the MAC line.

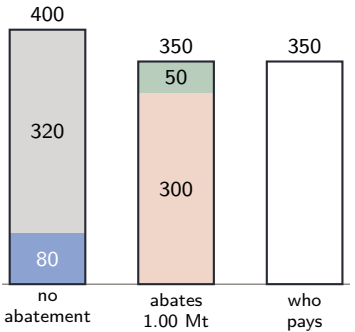
Differentiate $C(p) = pE - p^2/(2k)$ with respect to p. Which quantity in the picture does the slope equal?

Coal's net carbon cost at the first Orderly price

$$k = \bar{p}/E = 400/4.0 = 100$$
$$a^* = p/k = 100/100 = 1.00 \text{ Mt}$$
$$C = pE - \frac{p^2}{2k} = 400 - 50 = 350$$
$$N = (1 - \pi)\, C = 0.20 \times 350 =$$

Coal pays the carbon price on 3.00 Mt, a bill of USD 300 m, and abates 1.00 Mt at a cost of USD 50 m in 2026, the first year of the Orderly path.

Stylised values.



USD m. Blue: kept by Coal; grey: passed on; orange: bill on unabated tonnes; green: abatement.

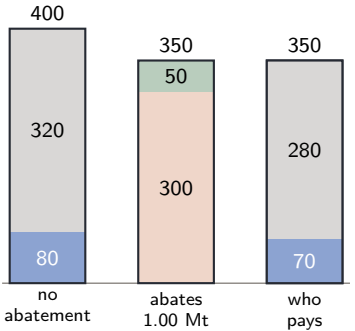
Repeat the four lines at the Orderly 2035 price of USD 300: what are Coal's abatement, compliance cost and net cost?

Coal's net carbon cost at the first Orderly price

$$k = \bar{p}/E = 400/4.0 = 100$$
$$a^* = p/k = 100/100 = 1.00 \text{ Mt}$$
$$C = pE - \frac{p^2}{2k} = 400 - 50 = 350$$
$$N = (1 - \pi)\, C = 0.20 \times 350 = 70$$

Coal pays the carbon price on 3.00 Mt, a bill of USD 300 m, and abates 1.00 Mt at a cost of USD 50 m in 2026, the first year of the Orderly path.

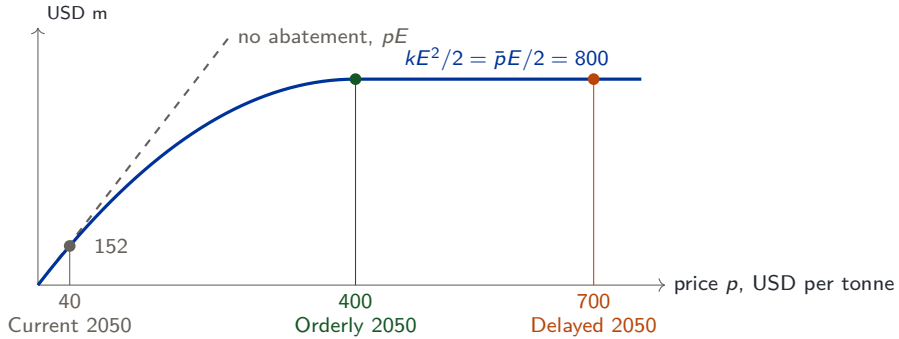
Stylised values.



USD m. Blue: kept by Coal; grey: passed on; orange: bill on unabated tonnes; green: abatement.

Repeat the four lines at the Orderly 2035 price of USD 300: what are Coal's abatement, compliance cost and net cost?

Above its full-abatement price a firm's compliance cost stops rising

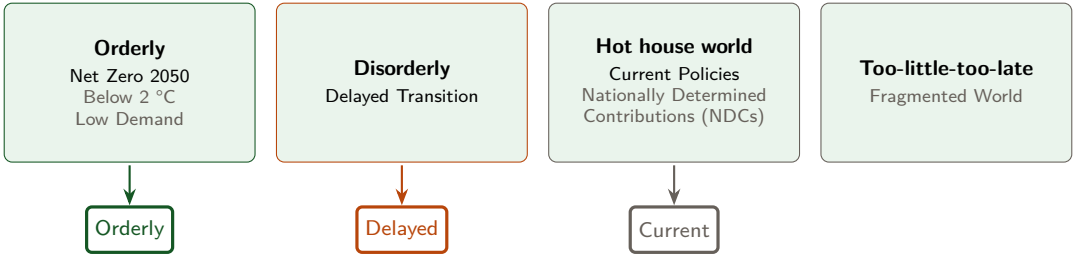


Above its full-abatement price, $p \geq \bar{p} = 400$, a firm abates all E and pays no carbon bill: Coal's compliance cost stays at USD 800 m whatever the price. The Delayed 2050 price of 700 costs Coal what the Orderly 400 costs it; without abatement it would cost USD 2,800 m. Stylised values.

What compliance cost does Cement, 3.0 Mt with full abatement at USD 1,200, pay at the Delayed 2050 price, and where does its cost stop rising?

The Orderly, Delayed and Current paths are shaped like three NGFS scenarios

1 2 3 4 **How does a scenario set a carbon price path?**



NGFS Phase V has four scenario families. Orderly is shaped like Net Zero 2050, Current like Current Policies, and Delayed like Delayed Transition, which adds no climate policy until 2030. Grey: no path follows.

NGFS (2024), long-term scenarios for central banks and supervisors, Phase V, p. 9; quadrant Disorderly, table p. 10, read 30 September 2026; families, p. 8.

What does the Delayed jump from USD 10 to 100 in 2031 do to Coal's net cost?

Between two points of a path the carbon price lies on a straight line

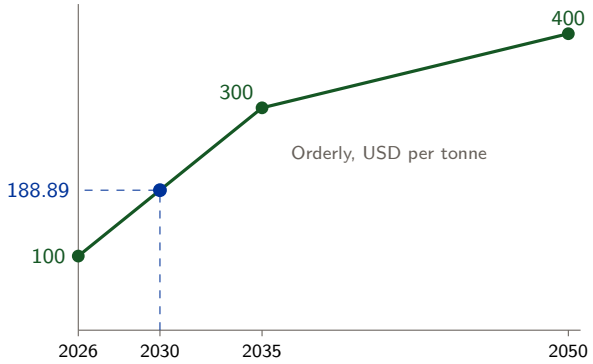
Each path of the six-firm book fixes the carbon price at a few years $a < b$; a year t between them gets the price on the straight line through the two points:

$$p_t = p_a + (p_b - p_a)\,\frac{t-a}{b-a}.$$

Orderly fixes 100 in 2026, 300 in 2035 and 400 in 2050. For 2030:

$$\begin{aligned} p_{2030} &= 100 + 200 \times \tfrac{4}{9} \\ &= 100 + 88.89 = 188.89. \end{aligned}$$

Stylised values.



What is the Delayed price in 2040, between its points of 200 in 2035 and 700 in 2050?

Constant-dollar carbon costs are discounted at the real rate of interest

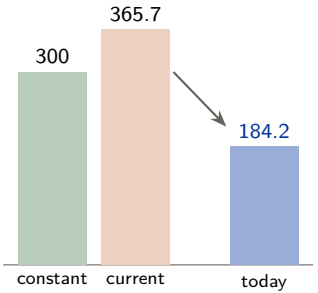
$$1 + i = (1 + r)(1 + j) \qquad \text{nominal } i\text{, inflation } j$$

$$\tilde{N}_t = N_t\,(1 + j)^t \qquad \text{constant to current dollars}$$

$$\frac{\tilde{N}_t}{(1 + i)^t} = \frac{N_t\,(1 + j)^t}{(1 + r)^t (1 + j)^t} \qquad \text{substitute } \tilde{N}_t \text{ and } 1 + i$$

$$= \frac{N_t}{(1 + r)^t} \qquad \text{cancel } (1 + j)^t$$

The book's prices are constant US dollars of 2025; $t = y - 2025$ counts years from the end of 2025. A constant-dollar net carbon cost N_t paid at the end of year y has the present value $N_t/(1 + r)^t$ at the end of 2025: the real rate of interest r discounts it.



Per Mt at the Orderly 2035 price, $t = 10$, $j = 2$, $r = 5$, $i = 7.1$ per cent:
$365.7/1.9856 = 300/1.6289 = 184.2$. Stylised values.

Inflation is 3 per cent a year and the nominal rate 6 per cent. Which real rate discounts the book's constant-dollar costs?

Hotelling path from the Orderly starting price

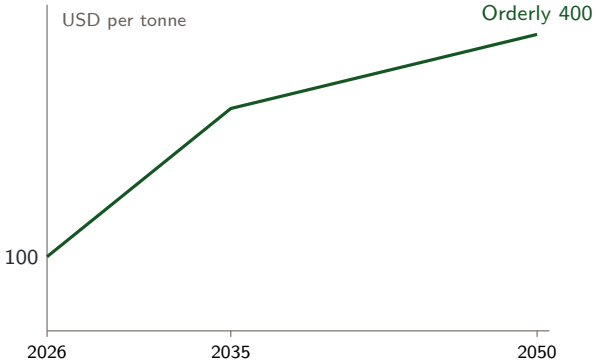
Hotelling (1931): the net price of an exhaustible resource, price less marginal extraction cost, rises at the rate of interest. A fixed carbon budget makes the right to emit such a resource:

$$p_y = p_{2026}\,(1+r)^{y-2026}.$$

From the Orderly starting price $p_{2026} = 100$, at the real rate $r = 5$ per cent, the Hotelling path over $y - 2026 = 24$ years reaches

$$p_{2050} = 100 \times 3.2251 =$$

Stylised values.



Hotelling, H. (1931), The Economics of Exhaustible Resources, Journal of Political Economy 39(2), 137 to 175, read 1 October 2026.

At what constant yearly rate would a price grow from USD 100 in 2026 to the Orderly 400 in 2050?

Hotelling path from the Orderly starting price

Hotelling (1931): the net price of an exhaustible resource, price less marginal extraction cost, rises at the rate of interest. A fixed carbon budget makes the right to emit such a resource:

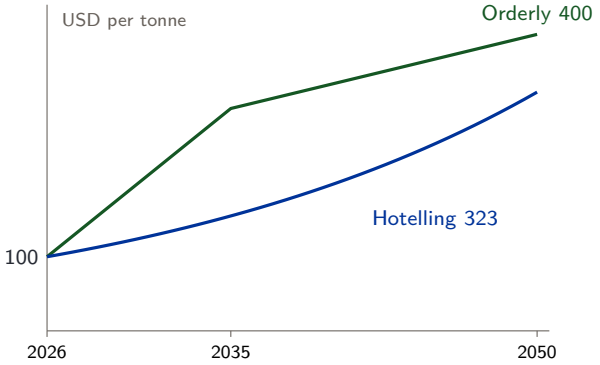
$$p_y = p_{2026}\,(1+r)^{y-2026}.$$

From the Orderly starting price $p_{2026} = 100$, at the real rate $r = 5$ per cent, the Hotelling path over $y - 2026 = 24$ years reaches

$$p_{2050} = 100 \times 3.2251 = 323$$

below the Orderly 400.

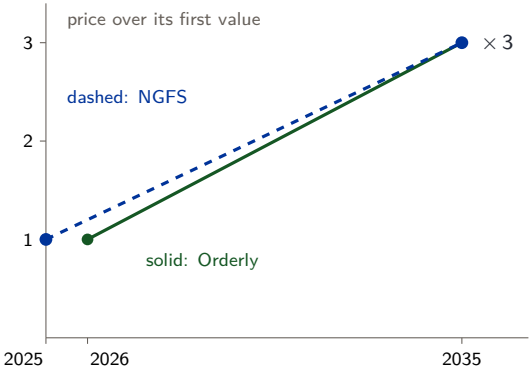
Stylised values.



Hotelling, H. (1931), The Economics of Exhaustible Resources, Journal of Political Economy 39(2), 137 to 175, read 1 October 2026.

At what constant yearly rate would a price grow from USD 100 in 2026 to the Orderly 400 in 2050?

The NGFS shadow carbon price triples in ten years, the Orderly path in nine



In NGFS Phase V, Net Zero 2050 needs a shadow carbon price of around 300 per tonne by 2035: from 98 to 294 within ten years, in US$2010: 3 times the starting price.

The Orderly path triples from 100 in 2026 to 300 in 2035, nine years, in constant US dollars of 2025. The units differ, so only the growth factors compare.

NGFS (2024), long-term scenarios for central banks and supervisors, Phase V, p. 21, REMIND-MAgPIE, weighted global, read 30 September 2026; the unit from the figure's axis label, read 1 October 2026.

At what yearly rate does the NGFS price grow from 98 to 294 in ten years?

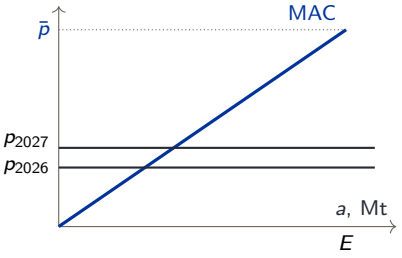
Abatement, net cost and present value of a paper mill

E	$\bar{p}$	π	p_{2026}	p_{2027}	r
2.0	500	0.60	150	200	5
Mt a year	USD per t	share	USD per t	USD per t	per cent

A paper mill emits E a year, abates fully at $\bar{p}$ and passes on the share π. Its carbon price is p_{2026} in 2026 and p_{2027} in 2027. In pairs, eight minutes:

1. the slope k and the abatement a^* in each year;
2. the compliance cost C and the net cost N in each year;
3. their present value at the end of 2025, each discounted at r with $t = y - 2025$.

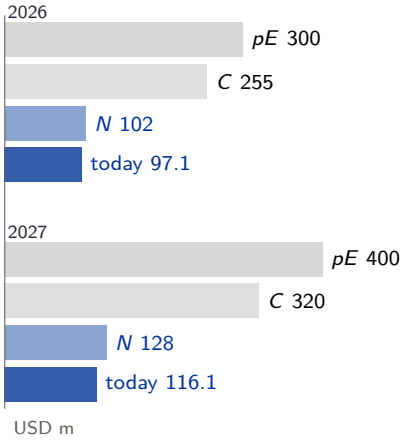
Stylised values.



Abatement lowers the paper mill's compliance cost below its carbon bill in both years

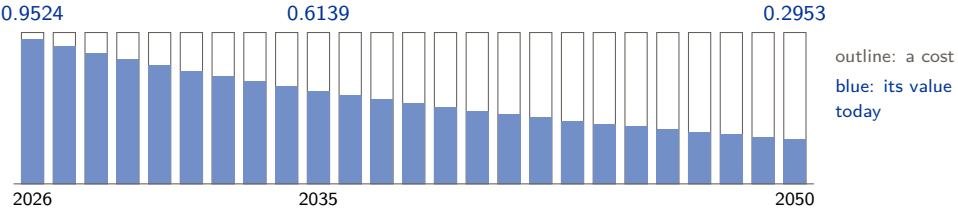
$$k = \bar{p}/E = 500/2.0 = 250$$
$$a^*_{2026} = 150/250 = 0.60 \text{ Mt}, \quad a^*_{2027} = 0.80 \text{ Mt}$$
$$C_{2026} = 300 - 45 = 255, \quad C_{2027} = 400 - 80 = 320$$
$$N_{2026} = 0.40 \times 255 = 102, \quad N_{2027} = 128$$
$$S = 102 \times 0.9524 + 128 \times 0.9070$$
$$= 97.1 + 116.1 = 213.2$$

Each net cost is multiplied by $1/(1+r)^t$, at $t = 1$ and $t = 2$. Abatement lowers the paper mill's compliance cost below its carbon bill pE in both years, and the two net costs, USD 102 m and USD 128 m, are worth USD 213.2 m at the end of 2025. Stylised values.



Each year of delay divides a cost's present value by one plus r

1 2 **3** 4 **What is a stream of carbon costs worth today?**



A net cost N_y paid at the end of year y has the present value $N_y/(1+r)^t$, $t = y - 2025$; the cost of 2026 is discounted once:

$$PV = \sum_{y=2026}^{2050} \frac{N_y}{(1+r)^{y-2025}}, \qquad d_t = \frac{1}{(1+r)^t}.$$

Each year of delay multiplies the discount factor d_t by $1/(1+r)$; at $r = 5$ per cent a cost of 2050, $t = 25$, counts 0.2953 of its value. Stylised values.

What is the 2040 discount factor at 5 per cent, and what is the Delayed 2040 price of 366.67 worth today per Mt?

A cost growing at g and discounted at r sums as a geometric series

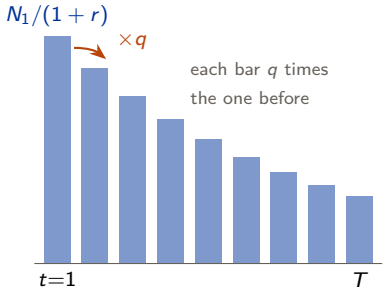
$$S = \sum_{t=1}^{T} \frac{N_1 (1+g)^{t-1}}{(1+r)^t}$$ first cost at $t = 1$, in 2026

$$= \frac{N_1}{1+r} \left(1 + q + q^2 + \cdots + q^{T-1}\right)$$ $q = (1+g)/(1+r)$

$$= \frac{N_1}{1+r} \cdot \frac{1 - q^T}{1 - q}$$ geometric sum

$$1 - q = \frac{r - g}{1+r}$$ common denominator

$$S = N_1 \frac{1 - q^T}{r - g}$$ cancel $1 + r$



T yearly net costs, the first N_1, each $1+g$ times the one before. The bracket is a geometric series of ratio q. The formula holds for any $g \neq r$; a cost that shrinks has $g < 0$.

Stylised positions.

What does S become when T grows without limit and $g < r$, and why must g stay below r?

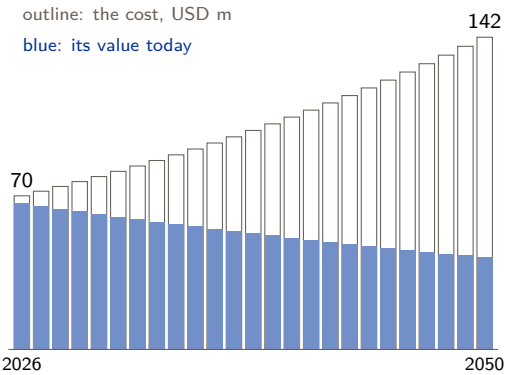
Growing annuity of Coal's first-year net cost

Coal's first-year net cost is $N_1 = \text{USD } 70$ m in 2026. Take $T = 25$ yearly costs, 2026 to 2050, each $g = 3$ per cent above the one before. The growing annuity at $r = 5$ per cent:

$$q = (1+g)/(1+r) = 0.980952$$
$$q^T = 0.6183, \qquad 1 - q^T = 0.3817$$
$$S = 70 \times 0.3817/0.02 = 1{,}336$$

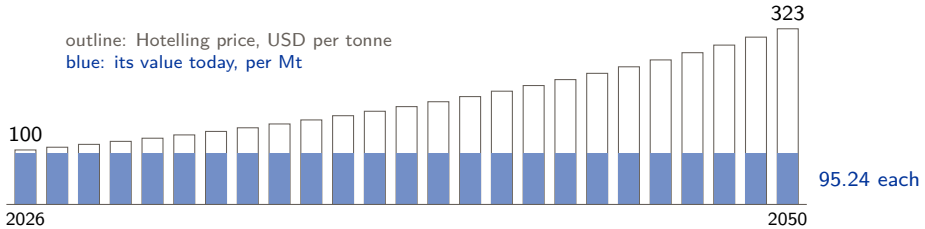
The cost reaches USD 142 m in 2050. The 25 costs sum to more than 25 times 70, and discounting brings them back to USD 1,336 m at the end of 2025.

Stylised values.



What are the 25 costs worth if Coal's net cost stays at USD 70 m a year, so that $g = 0$?

A price rising at the discount rate gives every year the same present value



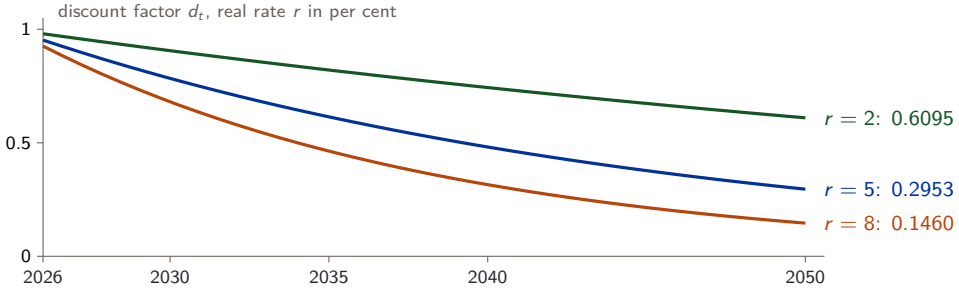
When the price rises at the discount rate, $g = r$, $q = 1$ and every term of the sum equals $p_{2026}/(1+r)$. For the Hotelling path from 100 at $r = 5$ per cent, each year is worth $100/(1+r) = 95.24$ today, and over $T = 25$ years

$$S = \frac{T\, p_{2026}}{1+r} = 25 \times 95.24 = 2{,}381, \text{ in USD m per Mt emitted a year.}$$

Stylised values.

The Hotelling price grows at 5 per cent, but the firm discounts at 3 per cent. By what factor does each year's present value change?

A lower real rate gives late costs more weight in present value



At 5 per cent a cost of 2050 counts 0.2953 of its value; at 2 per cent it counts 0.6095, 2.06 times as much, and at 8 per cent 0.1460. A cost of 2026 counts nearly the same at all three rates, so a lower rate gives late costs more weight. Stylised values.

By what factor does the present value of a cost paid at the end of 2045 grow when r falls from 5 to 2 per cent?

Present value of the three paths per Mt emitted a year

1 2 3 **4** **Which path costs a firm most once timing and discounting count?**

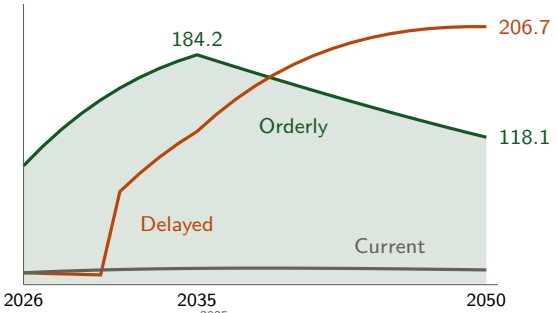
$$PV = \sum_{y=2026}^{2050} \frac{p_y}{(1+r)^{y-2025}}, \quad r = 5 \text{ per cent},$$

the present value per Mt emitted a year, in USD m:

path	price 2050	PV, USD m
Delayed	700	3,326
Orderly	400	3,689
Current	40	309

Orderly costs $3{,}689/3{,}326 - 1 = 10.9$ per cent more than Delayed. Its prices of 100 to 300 are discounted by 0.95 to 0.61; Delayed's of 2040 to 2050 by 0.48 to 0.30.

Stylised values.



price over $(1+r)^{y-2025}$, USD m per Mt;
shaded: the area that sums to Orderly's PV

What is the Orderly price of 333.33 in 2040 worth today per Mt, at 5 per cent?

Break-even real rate of the Orderly and Delayed paths

Per Mt emitted a year, 2026 to 2050:

	$r = 0$	$r = 5$
Orderly	7,300	3,689
Delayed	7,800	3,326

Undiscounted, Delayed costs more; at 5 per cent, Orderly does. The difference changes sign once in between, at the break-even real rate

$$r^* =$$

Stylised values.

At a real rate of 1 per cent, which path costs more per Mt, and which line of the chart says so?

Break-even real rate of the Orderly and Delayed paths

Per Mt emitted a year, 2026 to 2050:

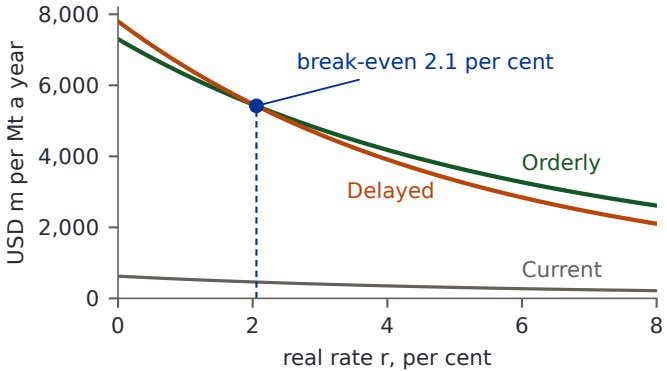
	$r = 0$	$r = 5$
Orderly	7,300	3,689
Delayed	7,800	3,326

Undiscounted, Delayed costs more; at 5 per cent, Orderly does. The difference changes sign once in between, at the break-even real rate

$$r^* = 2.1 \text{ per cent.}$$

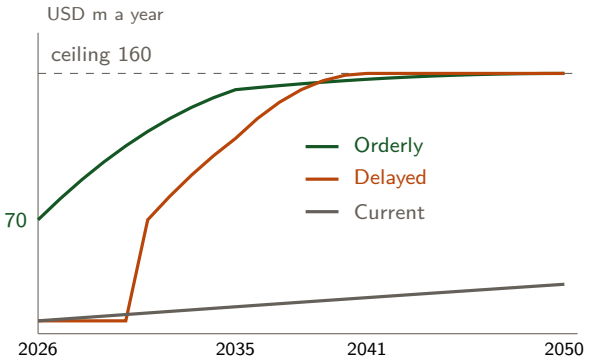
Below r^* Delayed costs a firm more per Mt.

Stylised values.



At a real rate of 1 per cent, which path costs more per Mt, and which line of the chart says so?

Coal's net cost stops rising once the price passes its full-abatement price



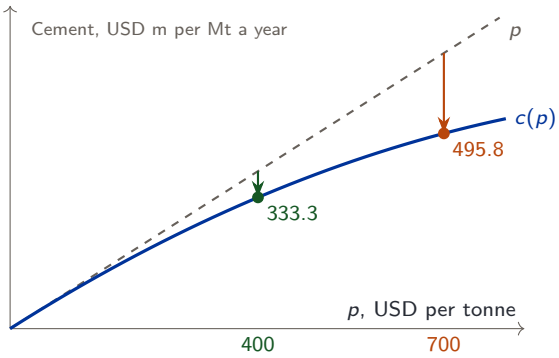
Coal's net cost $(1-\pi)\,C(p)$ stops rising at the ceiling $(1-\pi)\,\bar{p}E/2 = 160$ once the price passes its full-abatement price of 400: from 2041 on the Delayed path, only in 2050 on Orderly.

Present values at 5 per cent: Orderly USD 1,864 m, Delayed 1,342 m, Current 239 m. Orderly costs Coal 38.9 per cent more than Delayed, against 10.9 per cent per Mt without abatement.

Stylised values.

Undiscounted, Coal's costs are USD 3,514 m on Orderly and 2,860 m on Delayed. Is there a real rate of zero or above at which Delayed costs Coal more?

Abatement cuts the cost of Delayed's high late prices the most



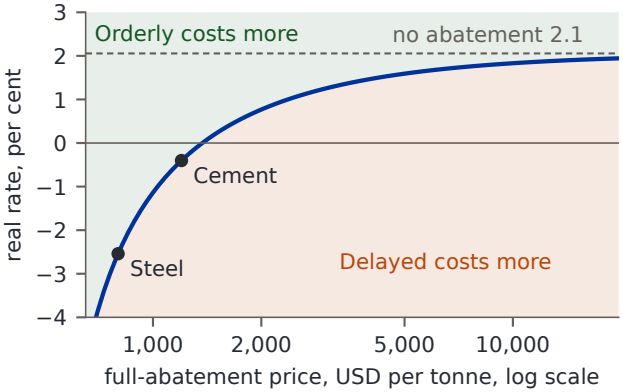
Per Mt, a firm pays $c(p) = p - p^2/(2\bar{p})$ while $p < \bar{p}$; Cement's $\bar{p}$ is 1,200. At the Orderly 2050 price it pays USD 333.3 m per Mt, 83.3 per cent of p; at the Delayed price, 495.8 m, 70.8 per cent.

Undiscounted, Cement's net cost is USD 1,899 m on Orderly and 1,878 m on Delayed: its break-even rate is −0.40 per cent.

Stylised values.

Steel abates fully at USD 800. What share of the Delayed 2050 price of 700 does its cost per Mt come to?

Coal, Steel and Cement pay more on Orderly at every non-negative rate



The cost per Mt, $c(p)$, depends on p and $\bar{p}$ alone, so one curve gives every firm's break-even rate: 2.1 per cent without abatement, zero at $\bar{p} = 1{,}371$.

Cement's is -0.40 per cent, Steel's -2.54, and Coal has none above -10 per cent: all three pay more on Orderly at every non-negative rate.

Stylised values.

A bank discounts its borrowers' carbon costs at 3 per cent real. For which full-abatement prices does Delayed cost a borrower more than Orderly?

Timing, discounting and abatement decide which carbon price path costs a firm most

1	How does a carbon price enter a firm's cash flow?	As the net cost $N = (1-\pi)(pE - p^2/(2k))$, capped at $(1-\pi)\,\bar{p}E/2$: Coal keeps USD 70 m at 100.
2	How does a scenario set a carbon price path?	The book's paths join a few points by straight lines, in constant US dollars of 2025; Hotelling reaches 323 by 2050.
3	What is a stream of carbon costs worth today?	The sum of $N_y/(1+r)^{y-2025}$; at growth g, $N_1(1-q^T)/(r-g)$.
4	Which path costs a firm most once timing and discounting count?	Per Mt at 5 per cent Orderly, 3,689 against 3,326; Delayed below 2.1 per cent. Full abatement above 10 and at most 1,371, as for Coal, Steel and Cement: Orderly at every rate of zero or above.

Stylised values.

Sources on Hotelling and the NGFS scenarios

- H. Hotelling (1931). The Economics of Exhaustible Resources. *Journal of Political Economy* 39(2), 137 to 175.

- NGFS (2024). NGFS long-term scenarios for central banks and supervisors, Phase V, November 2024: scenario families, p. 8; scenarios, pp. 9 and 10; shadow carbon price, p. 21, in US$2010 by the figure's axis label.

NGFS Phase V read on 30 September 2026, its p. 21 axis label and Hotelling on 1 October 2026.

Declarations

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Checkpoint: carbon costs, paths and present values

1. A firm emits 5.0 Mt, abates fully at USD 750, passes on 0.70. At a carbon price of USD 120 per tonne, find k, a^*, C and N.
2. The Delayed price in 2045; the Hotelling price in 2040 from 50 in 2026 at 4 per cent.
3. Nominal rate 6, inflation 2.5 per cent: the real rate?
4. Present value at the end of 2025 of USD 300 m paid at the end of 2040, at 4 per cent?
5. 10 yearly costs from USD 50 m in 2026, growing 2 per cent a year: present value at 6 per cent?
6. Full abatement at USD 300, $r = 1$ per cent: which of the Orderly and Delayed paths costs more per Mt, and why?

Stylised values.



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Carbon price paths and the present value of carbon costs - Mathematics of Green Finance Risk Management, Lecture 1

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The project site is https://digital-ai-finance.github.io/Green-Finance. A question about this file goes to joerg.osterrieder@utwente.nl.

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