Impermanent Loss: The Mathematics

Deriving the IL Formula from First Principles

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Learning Objectives

By the end of this lecture you will be able to:

1. **Define** impermanent loss precisely [Understand]
2. **Derive** the IL formula $\text{IL}(r) = \frac{2\sqrt{r}}{1+r} - 1$ from the CPMM invariant [Apply]
3. **Interpret** the formula at key price ratios [Analyze]
4. **Explain** when IL becomes permanent [Evaluate]

This deck derives a single formula from scratch. For broader DeFi context, see the DeFi Ecosystem deck.

Four objectives spanning Bloom's taxonomy from understanding to evaluation.

What Is Impermanent Loss?

Definition. Impermanent loss (IL) is the difference in value between:

(a) Holding tokens in a **liquidity pool** (LP position)

(b) Simply holding the same tokens in your **wallet** (HODL)

Key points:

- IL compares **LP position vs HODL**, not LP vs cash
- If price returns to its original level: $\text{IL} = 0$ (hence "impermanent")
- If you withdraw at a different price: loss is realized $\rightarrow$ **permanent**
- Also called **divergence loss** (more accurate term)

IL is not a loss compared to doing nothing — it is the opportunity cost of providing liquidity instead of holding.

Setup — The LP's Position

1. LP deposits x_0 ETH and y_0 USDC into a pool with invariant $x \cdot y = k$
2. Initial spot price: $p_0 = y_0/x_0$
3. After some time, external market price changes to p_1
4. Arbitrageurs trade until pool price matches p_1
5. New reserves: x_1, y_1 with $x_1 \cdot y_1 = k$ and $y_1/x_1 = p_1$

Question: Is the LP better or worse off compared to just holding x_0 ETH and y_0 USDC?

We assume zero fees and no concentrated liquidity (Uniswap V2 model). The invariant k stays constant.

Deriving the New Reserves

From $x_1 \cdot y_1 = k$ and $y_1 = p_1 \cdot x_1$:

$$x_1^2 \cdot p_1 = k \implies x_1 = \sqrt{k/p_1}, \qquad y_1 = \sqrt{k \cdot p_1}$$

Since $k = x_0 \cdot y_0 = x_0^2 \cdot p_0$:

$$x_1 = x_0\sqrt{p_0/p_1}, \qquad y_1 = y_0\sqrt{p_1/p_0}$$

Define the **price ratio** $r = p_1/p_0$:

$$\boxed{x_1 = \frac{x_0}{\sqrt{r}}, \qquad y_1 = y_0\sqrt{r}}$$

As r increases (price goes up), the pool holds less ETH and more USDC. Arbitrageurs buy the "cheap" ETH, pushing reserves toward the new equilibrium.

The new reserves are fully determined by the price ratio r and the initial position. This is the starting point for the IL formula.

Value of the LP Position vs HODL

Let $r = p_1/p_0$. Value everything in USDC at the new price p_1.

LP value:

$$V_{\mathrm{LP}} = x_1 \cdot p_1 + y_1 = \frac{x_0}{\sqrt{r}} \cdot r \cdot p_0 + y_0\sqrt{r}$$

Since $y_0 = x_0 \cdot p_0$:

$$V_{\mathrm{LP}} = x_0 p_0 \left(\sqrt{r} + \sqrt{r}\right) = 2\, x_0 p_0 \sqrt{r}$$

HODL value:

$$V_{\mathrm{HODL}} = x_0 \cdot p_1 + y_0 = x_0 p_0 (r + 1)$$

$$V_{\mathrm{LP}} = 2\, x_0 p_0 \sqrt{r} \qquad V_{\mathrm{HODL}} = x_0 p_0 (1 + r)$$

The LP value grows as $\sqrt{r}$ while HODL value grows linearly in r. This mismatch is the source of impermanent loss.

The IL Formula

$$\text{IL}(r) = \frac{V_{\text{LP}}}{V_{\text{HODL}}} - 1 = \frac{2\sqrt{r}}{1+r} - 1$$

$$\boxed{\text{IL}(r) = \frac{2\sqrt{r}}{1+r} - 1}$$

Properties:

- $\text{IL}(1) = 0$ (no price change $\rightarrow$ no loss)
- $\text{IL}(r) \leq 0$ for all $r > 0$ (always a loss or zero)
- $\text{IL}(r) = \text{IL}(1/r)$ (symmetric: $2\times$ up and $2\times$ down give the same IL)
- As $r \to 0$ or $r \to \infty$: $\text{IL} \to -1$ (total loss)

One formula captures all impermanent loss. The price ratio r is the only input needed.

Why Is IL Always Negative?

Claim: $2\sqrt{r} \leq 1 + r$ for all $r > 0$.

Proof:

$$\begin{aligned} (\sqrt{r} - 1)^2 &\geq 0 \quad \text{(square of a real number)} \\ r - 2\sqrt{r} + 1 &\geq 0 \\ 1 + r &\geq 2\sqrt{r} \end{aligned}$$

Equality holds if and only if $\sqrt{r} = 1$, i.e., $r = 1$ (price unchanged).

This is the **AM–GM inequality** applied to 1 and r:

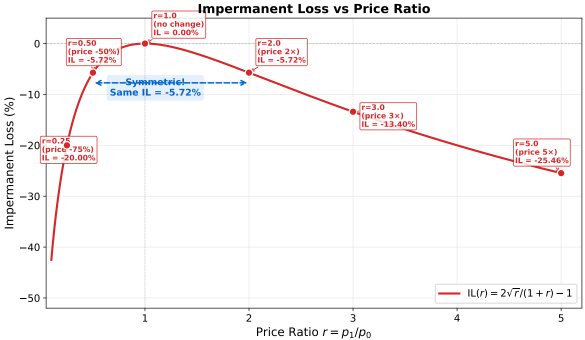
$$\text{AM} = \frac{1 + r}{2} \geq \sqrt{1 \cdot r} = \sqrt{r} = \text{GM}$$

Impermanent loss is a direct consequence of the arithmetic mean–geometric mean inequality.

The LP's portfolio is rebalanced along the curve $x \cdot y = k$ (geometric mean), but value is measured linearly (arithmetic mean). The gap between AM and GM is the IL.

IL at Key Price Ratios

Price change	r	IL formula	IL %
-75%	0.25	$2\sqrt{0.25}/1.25$	-20.00%
-50%	0.50	$2\sqrt{0.50}/1.50$	-5.72%
-25%	0.75	$2\sqrt{0.75}/1.75$	-1.03%
No change	1.00	$2/2$	0.00%
$+25\%$	1.25	$2\sqrt{1.25}/2.25$	-0.62%
$+50\%$	1.50	$2\sqrt{1.50}/2.50$	-2.02%
$+100\%$ ($2\times$)	2.00	$2\sqrt{2}/3$	-5.72%
$+200\%$ ($3\times$)	3.00	$2\sqrt{3}/4$	-13.40%
$+400\%$ ($5\times$)	5.00	$2\sqrt{5}/6$	-25.46%



Small price moves cause small IL ($\pm 25\%$ $\to$ about 1% IL). Large moves are devastating ($5\times$ $\to$ 25% IL).

IL Symmetry: $\text{IL}(r) = \text{IL}(1/r)$

A $2\times$ price **increase** and a 50% price **decrease** give the **same IL** (-5.72%).

Proof:

$$\frac{2\sqrt{1/r}}{1+1/r} = \frac{2/\sqrt{r}}{(r+1)/r} = \frac{2r}{(r+1)\sqrt{r}} = \frac{2\sqrt{r}}{1+r} \quad \checkmark$$

Upward move	**Equivalent downward move**
$r = 2$ (price doubles)	$r = 0.5$ (price halves)
$r = 3$ (price triples)	$r = 1/3$ (drops by 67%)
$r = 5$ (price $5\times$)	$r = 0.2$ (drops by 80%)

IL depends on **how far** the price moved from the starting point, not on the **direction**.

Symmetry follows algebraically from the ratio structure of the formula. The IL curve is symmetric around $r = 1$ on a log scale.

When IL Becomes Permanent

Impermanent (unrealized)

- Price returns to p_0 before withdrawal
- IL $= 0$ at that moment
- LP earned fees in the meantime
- Net result: **profitable**

Permanent (realized)

- LP withdraws at $p_1 \neq p_0$
- IL is locked in $\rightarrow$ **divergence loss**
- Cannot be “undone”
- Was it worth it? Only if fees $>$ IL

The term “impermanent” is misleading. It suggests the loss will disappear, but prices rarely return to exactly p_0. Many practitioners prefer **divergence loss**: the loss from the LP position diverging from a simple hold.

LP profitability requires: cumulative fee income $>$ IL at withdrawal time. High-volume, low-volatility pairs are most favorable.

IL vs Fee Income — When Is LP Profitable?

LP earns fees continuously.
IL grows with price deviation.

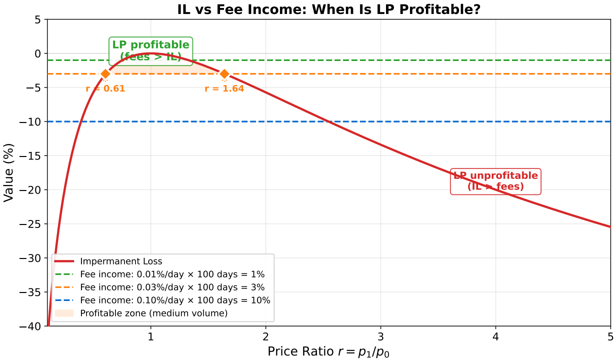
LP is profitable when:

$$\text{Fees} > |\text{IL}(r)|$$

For Uniswap V2 at 0.3% per trade:

- High-volume stable pair: very profitable
- Low-volume volatile pair: IL dominates

The chart shows IL (red curve) vs cumulative fee income at three daily rates over 100 days. Where IL is **above** the fee line, LP is profitable.



The break-even price ratio depends on fee income. Higher fees allow larger price deviations before IL overwhelms the position.

Key Takeaways

1. **One formula.** $\text{IL}(r) = 2\sqrt{r}/(1+r) - 1$ captures all impermanent loss for constant-product AMMs.

2. **Always ≤ 0.** IL is always a loss (or zero). This follows from the AM–GM inequality.

3. **Symmetric.** $\text{IL}(r) = \text{IL}(1/r)$. A 2× up and a 50% down give the same −5.72% IL.

4. **Zero only at $r = 1$.** IL vanishes only when price returns to its original level.

5. **Scale matters.** Small moves → small IL (±25% → ∼1%). Large moves → devastating (5× → 25%).

6. **Fees must beat IL.** LP is profitable only when cumulative fee income exceeds IL at the time of withdrawal.

The IL formula is the single most important equation for any liquidity provider to understand before committing capital.