Why Is the AMM Path-Independent?

The Mathematical Property Behind Trade Splitting

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Learning Objectives

By the end of this lecture you will be able to:

1. **Define** a state function and distinguish it from a path-dependent quantity [Understand]
2. **Prove** that the CPMM output is a state function using the telescoping sum [Apply]
3. **Connect** the telescoping sum to the fundamental theorem of calculus [Analyze]
4. **Explain** why trading fees break path independence and what changes algebraically [Analyze]

This deck is pure theory. For worked numerical examples, see the companion AMM Deep Dive deck.

Four objectives spanning Bloom's taxonomy from understanding to analysis.

The Question

In the companion deck, we showed numerically that selling 3×1 ETH gives the same USDC as selling 3 ETH at once.

But **why**?

Is this a coincidence, or is there a deep mathematical property at work?

This deck answers that question.
The answer is surprisingly simple.

The path independence property is the most important theoretical result about constant-function market makers.

State Function vs Path Function

State Function

- Depends only on current state
- Example: altitude of a hiker at position (x, y)
- Example: bank account balance today
- Example: temperature of a gas at (P, V)
- Change between two states: always the same, regardless of route

Path Function

- Depends on the route taken
- Example: total distance hiked
- Example: total fees paid over a year
- Example: heat absorbed by a gas
- Different routes $\rightarrow$ different totals

A **state function** is a quantity whose value is determined entirely by the current state of the system, independent of how the system reached that state.

State functions are central to thermodynamics, economics, and — as we will see — to AMMs.

In thermodynamics:

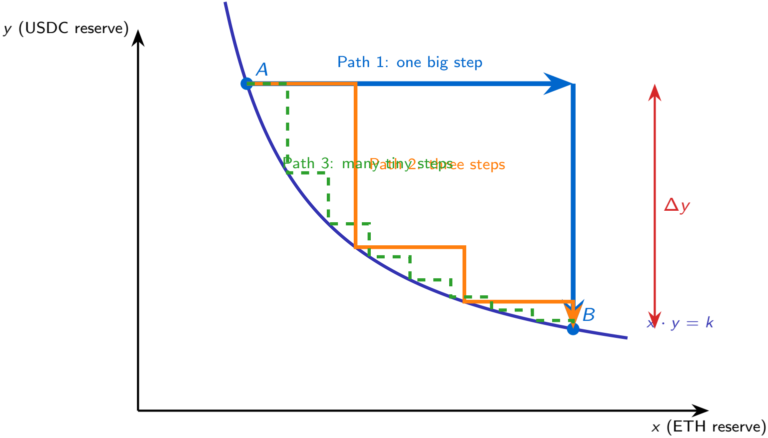
- State functions: internal energy U, entropy S, temperature T
- Path functions: heat Q, work W
- Going from state A to state B: ΔU is always the same; Q and W depend on the process

In AMMs:

- State: the reserves (x, y) with constraint $x \cdot y = k$
- State function: USDC reserve $y = k/x$
- Path-independent: total trade output $\Delta y = y(x_0) - y(x_0 + \Delta x)$
- Path-dependent (with fees): fee revenue (depends on number of sub-trades)

The AMM reserve y plays the role of internal energy: it depends only on the current state, not on history.

The AMM State Space



All paths yield $\Delta y = y(A) - y(B)$. The pool's reserves live on $x \cdot y = k$; total output depends only on start and end points, not on how many steps you take.

Three different trade-splitting strategies produce identical total output in the zero-fee case.

The Core Insight — y Is a Function of x Alone

The invariant $x \cdot y = k$ constrains the system to **one degree of freedom**:

$$\boxed{y = g(x) = \frac{k}{x}}$$

The output of any trade is the **difference** of this function at two points:

$$\Delta y_{\text{out}} = g(x_0) - g(x_0 + \Delta x)$$

This is a **function evaluation**, not a path integral.

Functions do not have paths — they have inputs and outputs.

That is the entire explanation.

Once you see that y is determined by x alone, path independence follows immediately. The rest is formalization.

Setting Up the Proof

Partition Δx into n arbitrary sub-trades $\delta_1, \delta_2, \ldots, \delta_n$ with $\sum_{i=1}^{n} \delta_i = \Delta x$.

Define intermediate reserves:

$$x_0, \quad x_1 = x_0 + \delta_1, \quad x_2 = x_1 + \delta_2, \quad \ldots, \quad x_n = x_0 + \Delta x$$

Each sub-trade's output:

$$\Delta y_i = g(x_{i-1}) - g(x_i) = \frac{k}{x_{i-1}} - \frac{k}{x_i}$$

(This is just: USDC reserve before trade i minus USDC reserve after trade i.)

Total output = sum of all sub-trade outputs:

$$\Delta y_{\text{total}} = \sum_{i=1}^{n} \left[g(x_{i-1}) - g(x_i) \right]$$

The partition is completely arbitrary: the δ_i can be unequal, and n can be any positive integer.

The Cancellation

Write out the sum for $n = 4$ to see the pattern:

$$\begin{aligned}
\Delta y_{\text{total}} &= [g(x_0) - \cancel{g(x_1)}] \\
&\quad + [\cancel{g(x_1)} - \cancel{g(x_2)}] \\
&\quad + [\cancel{g(x_2)} - \cancel{g(x_3)}] \\
&\quad + [\cancel{g(x_3)} - g(x_4)] \\
&= g(x_0) - g(x_n)
\end{aligned}$$

Every intermediate $g(x_i)$ appears once with $+$ and once with $-$. Only the **first** and **last** terms survive.

$g(x_0)$	$g(x_1)$	$g(x_2)$	$g(x_3)$	$g(x_4)$
✓	✗	✗	✗	✓

This is a telescoping sum — the same cancellation pattern that appears throughout mathematics.

The One-Line Proof

Theorem (Path Independence)

$$\sum_{i=1}^{n} [g(x_{i-1}) - g(x_i)] = g(x_0) - g(x_n)$$

for **any** partition $\delta_1, \ldots, \delta_n$ with $\sum \delta_i = \Delta x$. $\blacksquare$

The result depends only on:

- The **initial state** x_0 (where you start on the curve)
- The **total input** Δx (how far you move)
- The **invariant** k (which curve you are on)

It does **not** depend on: number of sub-trades, their sizes, or their order.

The entire proof is one line of algebra. Everything else in this deck is interpretation and context.

Discrete Telescoping = Fundamental Theorem of Calculus

Discrete (what we proved):

$$\sum_{i=1}^{n} \left[g(x_{i-1}) - g(x_i) \right] = g(x_0) - g(x_n)$$

Continuous (FTC):

$$\int_{x_0}^{x_n} g'(x)\, dx = g(x_n) - g(x_0)$$

These are the **same idea**. The discrete telescoping sum is the discrete analogue of the fundamental theorem of calculus.

The AMM path independence proof is simply the FTC applied to $g(x) = k/x$.

Telescoping sums and the FTC both say: the total change of a function equals the difference of its boundary values.

The Infinitesimal Trade

In the continuous limit, an infinitesimal deposit of dx ETH yields:

$$dy_{\text{out}} = -dg = -g'(x)\,dx = \frac{k}{x^2}\,dx$$

Integrating from x_0 to $x_0 + \Delta x$:

$$\Delta y_{\text{out}} = \int_{x_0}^{x_0+\Delta x} \frac{k}{x^2}\,dx = \left[-\frac{k}{x}\right]_{x_0}^{x_0+\Delta x}$$

$$= \frac{k}{x_0} - \frac{k}{x_0 + \Delta x}$$

Same result. The integral depends only on the limits of integration — the FTC guarantees path independence.

The derivative $g'(x) = -k/x^2$ is the instantaneous exchange rate: how much USDC per infinitesimal ETH at reserve level x.

Exact Differentials

The form $dy_{\text{out}} = (k/x^2)\,dx$ is an **exact differential**: it is the total differential of a function (namely $-k/x$).

Key theorem. A differential form $P(x)\,dx$ is **exact** if and only if $\int P\,dx$ is path-independent.

For the AMM: $P(x) = k/x^2$ has antiderivative $-k/x = -y$.

So the integral (total trade output) is always path-independent. No special property of $x \cdot y = k$ is needed beyond the fact that y is a function of x alone.

Students who have taken multivariable calculus may recognize this as the statement that the "exchange rate field" $F(x) = k/x^2$ is conservative ($F = -\nabla\phi$ with $\phi = -k/x$). The telescoping proof on Slide 9 is complete without this.

Exactness is the calculus formalization of why function differences are path-independent.

Fees Break Path Independence

With a fee fraction f (e.g., $f = 0.003$ in Uniswap V2):

1. The **effective input** is $\delta_{i,\text{eff}} = \delta_i(1 - f)$
2. But the **full** δ_i enters reserves: $x_i = x_{i-1} + \delta_i$
3. So the new invariant: $k_i = x_i \cdot y_i > k_{i-1}$ (k **grows**!)

The invariant is no longer constant between trades. Each sub-trade operates on a **different curve**.

The fee creates a wedge between the effective input (used for pricing) and the actual input (added to reserves), causing k to increase.

Why Telescoping Fails

With fees, the function changes between trades: $g_i(x) = k_i/x$ where $k_1 < k_2 < \cdots < k_n$.

The sum becomes:

$$\sum_{i=1}^{n} [g_i(x_{i-1}) - g_i(x_i)]$$

But $g_1(x_1) \neq g_2(x_1)$ because $k_1 \neq k_2$.

Trade 1 produces $-g_1(x_1)$. Trade 2 starts with $+g_2(x_1)$. These do **not** cancel:

$$\cancel{g_1(x_1)} \neq g_2(x_1) \quad \text{because } k_1 \neq k_2$$

Telescoping requires the **same function** g evaluated at consecutive points. Fees change g between trades, breaking the cancellation.

The subscript on g_i is what kills path independence: each sub-trade lives on a different hyperbola.

Three Ways to Break Path Independence

1. **Trading fees** (k grows after each sub-trade)
   Each sub-trade pays a separate fee, so splitting a trade costs more in total fees
2. **External state changes**
   If another trader modifies reserves between your sub-trades, the state (x, y) jumps to a different point, changing your starting position
3. **Time-dependent invariants**
   If k changes over time (e.g., Balancer LBP auctions), the curve itself shifts between trades

Common thread: path independence requires the invariant to be **constant** across all sub-trades.

In practice, all three conditions are violated simultaneously: every real DEX has fees, concurrent traders, and block-by-block state updates.

Beyond $x \cdot y = k$

The path independence property is not special to the constant product formula. It holds for **any** invariant.

Theorem. Let $f(x, y) = C$ be any smooth invariant that implicitly defines $y = g(x)$. Then the total output for a trade of Δx is:

$$\Delta y = g(x_0) - g(x_0 + \Delta x)$$

which is path-independent by the same telescoping argument.

AMM Type	**Invariant**	$g(x)$
Constant Product	$x \cdot y = k$	k/x
Constant Sum	$x + y = C$	$C - x$ (linear)
Weighted	$x^w \cdot y^{1-w} = k$	$(k/x^w)^{1/(1-w)}$

All are path-independent in the no-fee case.

The key requirement is not the specific formula but the existence of a smooth invariant $f(x, y) = C$ defining y as a function of x.

The General Principle

Theorem (Path Independence of CFMMs).

Any constant-function market maker with invariant $f(x, y) = C$, where:

1. the fee is zero,
2. no external state changes occur between sub-trades, and
3. f is smooth enough to define $y = g(x)$ implicitly,

is path-independent. The total trade output depends only on the initial reserves and the total trade size.

Proof sketch: $f(x, y) = C$ defines $y = g(x)$. Output $= g(x_0) - g(x_0 + \Delta x)$. Telescoping. ■

This theorem covers Uniswap (constant product), Curve (StableSwap), Balancer (weighted geometric mean), and any future CFMM design.

The Answer in One Slide

1. The invariant $x \cdot y = k$ constrains the pool to **one degree of freedom**: $y = k/x$
2. Trade output = difference of this function: $\Delta y = g(x_0) - g(x_0 + \Delta x)$
3. Differences **telescope**: $\sum [g(x_{i-1}) - g(x_i)] = g(x_0) - g(x_n)$
4. Telescoping = the discrete **fundamental theorem of calculus**
5. Fees break it because they change g between trades (k grows)

Five sentences that capture the entire theoretical argument from motivation to limitation.

Key Takeaways

1. Path independence = state function. The output depends only on initial reserves and total trade size, not on splitting or ordering.

2. The reason is algebraic. $y = k/x$ is a function of x alone. Function differences telescope. That is the whole proof.

3. It generalizes. Any CFMM with a smooth invariant $f(x, y) = C$ is path-independent (no-fee).

4. Fees are the exception. Fees change k between trades, breaking the telescoping cancellation. With fees, splitting trades changes the total cost.

The mathematical simplicity of this result is what makes constant-function market makers so elegant and analyzable.

Appendix

Technical Details and Further Reading

Formal Definition — State Function

A function $F : \mathcal{S} \to \mathbb{R}$ on a state space $\mathcal{S}$ is a **state function** iff for any two paths γ_1, γ_2 from state A to state B:

$$\Delta F = F(B) - F(A) \quad \text{(independent of path)}$$

Equivalently, for any closed path γ (starting and ending at the same state):

$$\oint_\gamma dF = 0$$

In AMM terms: if you execute trades that return the pool to its original reserves, the net USDC transferred is zero.

The closed-path characterization provides a practical test: any round-trip trade in a zero-fee AMM has zero net output.

Fee Counterexample — Algebraic Detail

Two trades of δ each vs one trade of 2δ, with fee f.

Sequential (two trades):

$$\text{Trade 1:} \quad \Delta y_1 = \frac{k_0 \cdot (1-f)\delta}{x_0 + (1-f)\delta}$$

$$k_1 = (x_0 + \delta)(y_0 - \Delta y_1) > k_0$$

$$\text{Trade 2:} \quad \Delta y_2 = \frac{k_1 \cdot (1-f)\delta}{x_1 + (1-f)\delta} \neq \frac{k_0 \cdot (1-f)\delta}{x_1 + (1-f)\delta}$$

Single trade:

$$\Delta y = \frac{k_0 \cdot (1-f) \cdot 2\delta}{x_0 + (1-f) \cdot 2\delta}$$

Since $k_1 > k_0$, Trade 2 of the sequential path produces more USDC than it would with k_0. So $\Delta y_1 + \Delta y_2 > \Delta y$ — splitting gives **more** output with fees (each sub-trade sees a richer pool).

This is the opposite of what some students expect! Fees make splitting advantageous, not disadvantageous.

The direction of the fee effect is counterintuitive: more sub-trades means more total fee paid but also more k-growth benefiting later trades.

Reading List

Primary references:

- Angeris & Chitra, "Improved Price Oracles: Constant Function Market Makers" (2020)
- Angeris et al., "An analysis of Uniswap markets" (2019, arXiv:1911.03380)
- Xu et al., "SoK: DEX with AMM protocols" (ACM CSUR, 2023)

Companion material:

- AMM Deep Dive deck: `lectures/amm_deep_dive/`
  Worked numerical examples demonstrating path independence with concrete ETH/USDC values

The Angeris & Chitra paper provides the most rigorous treatment of CFMM properties including path independence.