The Constant Product Formula: $x \cdot y = k$

Ultra-Deep Math Walkthrough with Worked Examples

Prof. Dr. Jörg Osterrieder

BSc Blockchain, Crypto Economy & NFTs

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Learning Objectives

After this lecture you will be able to:

1. **Derive** the constant-product trade formula $\Delta y_{\text{out}} = \dfrac{y_0 \cdot \Delta x}{x_0 + \Delta x}$ from the invariant $x \cdot y = k$ [Apply]
2. **Compute** execution price, spot price, slippage, and price impact for any ETH/USDC trade [Apply]
3. **Distinguish** slippage from price impact and explain why they are not the same concept [Analyze]
4. **Prove** that splitting a trade into smaller pieces yields the same total output (path independence) [Analyze]

A note on style

This is a math-forward deck. Formulas appear on every slide. A companion deck covers the same ideas without math.

Bloom taxonomy tags in brackets. All derivations use the primary pool $x_0 = 1\,000$ ETH, $y_0 = 3\,000\,000$ USDC.

Prerequisites

Liquidity pool — a smart contract holding two token reserves (x, y) that enables trading without an order book.

USDC — a USD-pegged stablecoin; treat 1 USDC $\approx$ \$1 throughout this deck.

Spot (marginal) price — the instantaneous exchange rate at current reserves: $p = y/x$.

Execution price — the average price actually paid for a finite trade: $p_{\text{exec}} = \Delta y_{\text{out}}/\Delta x$.

Percent change — $\dfrac{\text{new} - \text{old}}{\text{old}} \times 100$.

Basic algebra — solving for x in $x \cdot y = k$, i.e. $x = k/y$.

If any term above is unfamiliar, review the introductory DeFi deck before continuing.

The Problem

You want to **sell 10 ETH for USDC**.

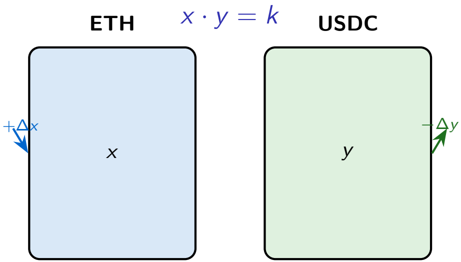
There is no order book and no broker. The pool holds x ETH and y USDC.

Central question

How many USDC will you receive?

The pool enforces exactly one rule:

$$x \cdot y = k \quad \text{(constant during a trade)}$$

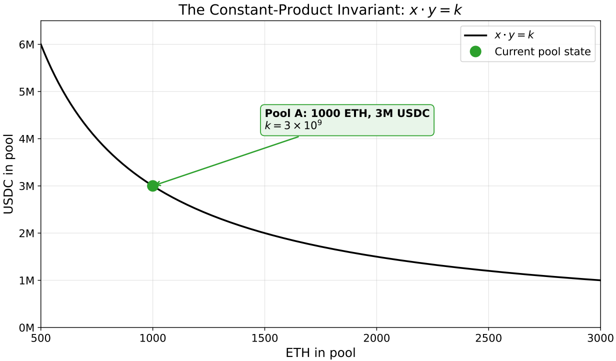


We answer this in exact numbers by slide 12.

The Invariant: $x \cdot y = k$

$$\boxed{x \cdot y \;=\; k \quad \text{(constant during a trade)}}$$

- x = reserve of ETH in the pool
- y = reserve of USDC in the pool
- k = positive constant fixed by current reserves
- Every valid trade moves along the hyperbola $xy = k$



The curve $xy = k$ is a rectangular hyperbola. As $x \to \infty$, $y \to 0$ but never reaches zero — you can never drain one side completely.

Primary Example Pool

Pool A — used in every worked example unless stated otherwise

$$
\begin{aligned}
x_0 &= 1\,000 \text{ ETH} \\
y_0 &= 3\,000\,000 \text{ USDC} \\
k &= x_0 \cdot y_0 = 1\,000 \times 3\,000\,000 = 3\,000\,000\,000 = 3 \times 10^9 \\
p_0 &= \frac{y_0}{x_0} = \frac{3\,000\,000}{1\,000} = 3\,000 \text{ USDC/ETH} \\
\text{Pool TVL} &= 2 \cdot y_0 = 6\,000\,000 \text{ USDC} \;\approx\; \$6\text{M}
\end{aligned}
$$

Every example in this deck uses Pool A. The only exception is slide 20, which introduces a smaller Pool B for comparison.

TVL (Total Value Locked) doubles the USDC side because the pool is balanced: $x_0 \cdot p_0 = y_0$, so each side is worth y_0 in dollar terms.

Deriving the Spot Price

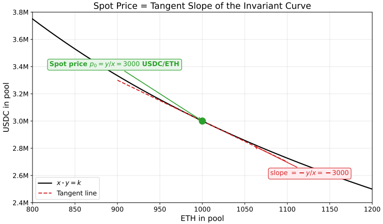
Start from the invariant and differentiate:

$$y = \frac{k}{x}$$

$$\frac{dy}{dx} = -\frac{k}{x^2} = -\frac{y}{x}$$

$$\text{Marginal (spot) price:} \quad p = -\frac{dy}{dx} = \frac{y}{x}$$

$$\text{Plug in Pool A:} \quad p_0 = \frac{3\,000\,000}{1\,000} = 3\,000 \text{ USDC/ETH}$$



The spot price is the slope of the tangent to the hyperbola. The negative sign cancels because selling ETH ($+\Delta x$) removes USDC ($-\Delta y$). → Appendix.

The Trade Formula

A trader deposits Δx ETH into the pool. Derive the USDC received:

$$x_1 = x_0 + \Delta x$$

$$y_1 = \frac{k}{x_1} = \frac{x_0 \cdot y_0}{x_0 + \Delta x}$$

$$\Delta y_{\text{out}} = y_0 - y_1 = y_0 - \frac{x_0 \cdot y_0}{x_0 + \Delta x} = y_0 \left(1 - \frac{x_0}{x_0 + \Delta x}\right)$$

$$\boxed{\Delta y_{\text{out}} = \frac{y_0 \cdot \Delta x}{x_0 + \Delta x}}$$

This single formula answers every "how much do I get?" question in this deck.

Key insight: Δy_{out} grows with Δx but is always less than y_0. As $\Delta x \to \infty$, $\Delta y_{\text{out}} \to y_0$.

Execution Price vs Spot Price

Three prices to distinguish:

$$\text{Spot price before:} \quad p_0 = \frac{y_0}{x_0}$$

$$\text{Execution price:} \quad p_{\text{exec}} = \frac{\Delta y_{\text{out}}}{\Delta x} = \frac{y_0}{x_0 + \Delta x} \quad \text{(divide trade formula by } \Delta x\text{)}$$

$$\text{Spot price after:} \quad p_1 = \frac{y_1}{x_1} = \frac{k}{x_1^2} = p_0 \cdot \left(\frac{x_0}{x_1}\right)^2$$

Ordering for selling ETH ($\Delta x > 0$):

$$p_1 \; < \; p_{\text{exec}} \; < \; p_0$$

For buying ETH (depositing USDC), the inequality reverses: $p_{\text{exec}} > p_0$.

The execution price is the average rate over the trade. It always lies between the old and new spot prices.

Worked Example 1 — Sell $\Delta x = 1$ ETH

$$x_1 = 1\,000 + 1 = 1\,001$$

$$y_1 = \frac{3\,000\,000\,000}{1\,001} = 2\,997\,002.9970 \text{ USDC}$$

$$\Delta y_{\text{out}} = 3\,000\,000 - 2\,997\,002.9970 = 2\,997.0030 \text{ USDC}$$

$$p_{\text{exec}} = \frac{2\,997.0030}{1} = 2\,997.0030 \text{ USDC/ETH}$$

$$\text{Slippage} = \frac{\Delta x}{x_0 + \Delta x} = \frac{1}{1\,001} = 0.0999\,\%$$

$$\text{Price impact} = 1 - \left(\frac{1\,000}{1\,001}\right)^2 = 0.1996\,\%$$

Tiny trade → slippage ≈ 0.1 %, execution price ≈ spot price.

Slippage = $0.0999\,\%$ **vs. price impact =** $0.1996\,\%$**. The ratio is** $\approx 2.00\times$ **— the small-trade limit.**

Worked Example 2 — Sell $\Delta x = 10$ ETH

$$
\begin{aligned}
x_1 &= 1\,000 + 10 = 1\,010 \\
y_1 &= \frac{3\,000\,000\,000}{1\,010} = 2\,970\,297.0297 \text{ USDC} \\
\Delta y_{\text{out}} &= 3\,000\,000 - 2\,970\,297.0297 = 29\,702.9703 \text{ USDC} \\
p_{\text{exec}} &= \frac{29\,702.9703}{10} = 2\,970.2970 \text{ USDC/ETH} \\
\text{Slippage} &= \frac{10}{1\,010} = 0.9901\,\% \\
\text{Price impact} &= 1 - \left(\frac{1\,000}{1\,010}\right)^2 = 1.9704\,\%
\end{aligned}
$$

Price impact (1.97 %) is about **twice** the slippage (0.99 %). This is the small-trade limit at work.

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10 **ETH is** 1 % **of the pool. Verify:** $1\,010 \times 2\,970\,297.0297 = 3\,000\,000\,000$ ✓

Worked Example 3 — Sell $\Delta x = 100$ ETH (10% of pool)

$$
\begin{aligned}
x_1 &= 1\,000 + 100 = 1\,100 \\
y_1 &= \frac{3\,000\,000\,000}{1\,100} = 2\,727\,272.7273 \text{ USDC} \\
\Delta y_{\text{out}} &= 3\,000\,000 - 2\,727\,272.7273 = 272\,727.2727 \text{ USDC} \\
p_{\text{exec}} &= \frac{272\,727.2727}{100} = 2\,727.2727 \text{ USDC/ETH} \\
\text{Slippage} &= \frac{100}{1\,100} = 9.0909\,\% \\
\text{Price impact} &= 1 - \left(\frac{1\,000}{1\,100}\right)^2 = 17.3554\,\%
\end{aligned}
$$

$10\,\%$ **of pool** $\rightarrow \sim 9\,\%$ slippage. Ratio PI/Slippage $= 1.91$, no longer exactly $2\times$.

At this size the second-order terms matter: slippage $\approx \varepsilon - \varepsilon^2$, price impact $\approx 2\varepsilon - \varepsilon^2$ where $\varepsilon = 0.1$.

Worked Example 4 — Sell $\Delta x = 500$ ETH (50% of pool)

$$x_1 = 1\,000 + 500 = 1\,500$$

$$y_1 = \frac{3\,000\,000\,000}{1\,500} = 2\,000\,000 \text{ USDC}$$

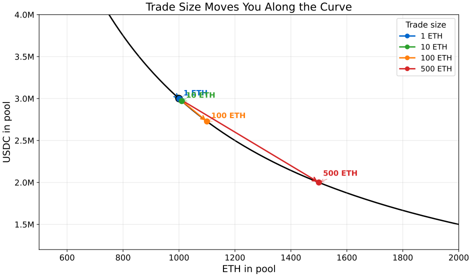
$$\Delta y_{\text{out}} = 3\,000\,000 - 2\,000\,000 = 1\,000\,000 \text{ USDC}$$

$$p_{\text{exec}} = \frac{1\,000\,000}{500} = 2\,000 \text{ USDC/ETH}$$

$$\text{Slippage} = \frac{500}{1\,500} = 33.33\,\%$$

$$\text{Price impact} = 1 - \left(\frac{1\,000}{1\,500}\right)^2 = 55.56\,\%$$

Selling **half the pool's ETH** collapses the spot price by more than half.



New spot $p_1 = 3\,000\,000\,000/1\,500^2 = 1\,333.33$ **USDC/ETH, i.e.** $-55.56\,\%$ **from** $p_0 = 3\,000$.

Comparison of All Four Trades

Δx (ETH)	x_1	y_1 (USDC)	Δy_{out} (USDC)	p_{exec}	Slippage	Price Impact
1	1 001	2 997 003.00	2 997.00	2 997.00	0.10 %	0.20 %
10	1 010	2 970 297.03	29 702.97	2 970.30	0.99 %	1.97 %
100	1 100	2 727 272.73	272 727.27	2 727.27	9.09 %	17.36 %
500	1 500	2 000 000.00	1 000 000.00	2 000.00	33.33 %	55.56 %

Key patterns visible in the table:

- Slippage grows roughly linearly with $\Delta x / x_0$ for small trades
- Price impact is always larger than slippage (ratio starts at $2\times$ and decreases)
- Δy_{out} is always less than $\Delta x \cdot p_0$ (you always get less than the quoted price)

Every row satisfies $x_1 \cdot y_1 = 3\,000\,000\,000$ exactly. Values rounded to 2 dp for readability; full precision on individual slides.

Slippage vs Price Impact — Two Different Things

Slippage

"How much worse was my average price vs the quoted spot?"

Price Impact

"How much did my trade shift the pool's spot price?"

$$\begin{aligned}\text{Slippage} &= 1 - \frac{p_{\text{exec}}}{p_0} \\ &= 1 - \frac{y_0/(x_0 + \Delta x)}{y_0/x_0} \\ &= \frac{\Delta x}{x_0 + \Delta x}\end{aligned}$$

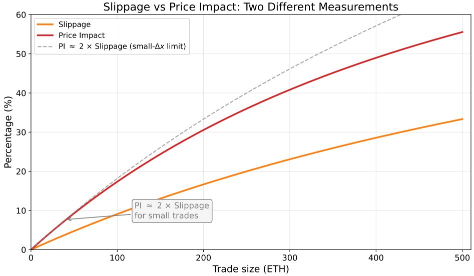
$$\begin{aligned}\text{PI} &= 1 - \frac{p_1}{p_0} \\ &= 1 - \left(\frac{x_0}{x_0 + \Delta x}\right)^2\end{aligned}$$

Key result: Price impact $\approx 2 \times$ Slippage (for small $\Delta x \ll x_0$), and PI $\geq$ Slippage always.

Slippage is about YOUR trade. Price impact is about what your trade does to THE POOL. → Closed-form derivation in Appendix.

Slippage, Price Impact, and Their Ratio

Δx (ETH)	Slippage	Price Impact	Ratio PI / Slip
1	0.10 %	0.20 %	2.00
10	0.99 %	1.97 %	1.99
100	9.09 %	17.36 %	1.91
500	33.33 %	55.56 %	1.67
1 000	50.00 %	75.00 %	1.50
2 000	66.67 %	88.89 %	1.33



Ratio $\to 2.00$ **as** $\Delta x \to 0$**; decreases toward** 1 **as** $\Delta x \to \infty$**.** $\to$ **Taylor expansion in Appendix.**

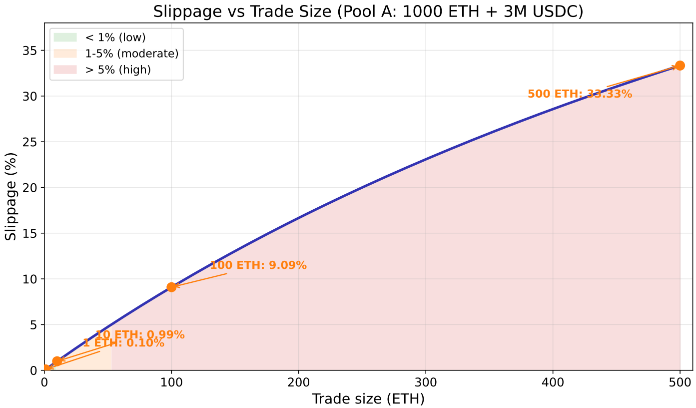
Slippage vs Trade Size

The curve shows slippage as a function of Δx:

$$s(\Delta x) = \frac{\Delta x}{x_0 + \Delta x}$$

Properties:

- $s(0) = 0$
- $s(x_0) = 50\,\%$
- $\lim\limits_{\Delta x \to \infty} s = 100\,\%$
- Concave: diminishing marginal slippage



For Pool A ($x_0 = 1\,000$ ETH), a trade equal to the entire reserve ($\Delta x = 1\,000$) incurs exactly $50\,\%$ slippage.

Does Splitting a Trade Help? (Path Independence)

Question: If I sell 3 ETH, do I get more USDC by splitting into three 1-ETH trades?

Answer

No. In a CPMM with no trading costs, splitting makes no difference. The total USDC received is **identical**.

Why: The final pool state depends only on the total Δx deposited, not on how many sub-trades you use:

$$y_1 = \frac{k}{x_0 + \Delta x_{\text{total}}} \quad \Longrightarrow \quad \Delta y_{\text{total}} = y_0 - \frac{k}{x_0 + \Delta x_{\text{total}}}$$

This expression depends on Δx_{total} only — not on the number of steps.

Path independence is a direct consequence of the invariant $x \cdot y = k$ being state-dependent, not path-dependent. We verify this numerically on the next slide. → Formal proof in Appendix.

Worked Example — Three 1-ETH Trades vs One 3-ETH Trade

Path A: Three sequential 1-ETH trades

Trade 1 into $(1\,000,\ 3\,000\,000)$:

$$\Delta y_1 = \frac{3\,000\,000 \times 1}{1\,001} = 2\,997.0030$$

Trade 2 into $(1\,001,\ 2\,997\,002.997)$:

$$\Delta y_2 = \frac{2\,997\,002.997 \times 1}{1\,002} = 2\,991.0210$$

Trade 3 into $(1\,002,\ 2\,994\,011.976)$:

$$\Delta y_3 = \frac{2\,994\,011.976 \times 1}{1\,003} = 2\,985.0568$$

Total:

$$2\,997.0030 + 2\,991.0210 + 2\,985.0568$$
$$= \mathbf{8\,973.0808}\ \text{USDC}$$

Path B: One 3-ETH trade

$$\Delta y = \frac{3\,000\,000 \times 3}{1\,003}$$
$$= \mathbf{8\,973.0808}\ \text{USDC}$$

Result

Identical.
The path does not matter.

Both paths end at the same point on the hyperbola: $(1\,003,\ 2\,991\,026.92)$**. The route is different; the destination is the same.**

Pool Depth Matters — Same Trade, Different Pools

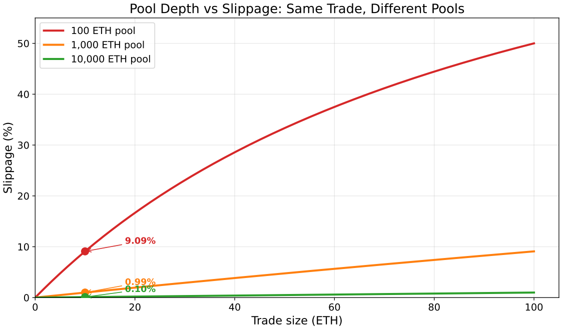
Pool B (small): $x_0 = 100$, $y_0 = 300\,000$, $k = 3 \times 10^7$, $p_0 = 3\,000$.

Pool A (large): $x_0 = 1\,000$, $y_0 = 3\,000\,000$, $k = 3 \times 10^9$, $p_0 = 3\,000$.

Trade: sell $\Delta x = 10$ ETH in both.

	Pool B	**Pool A**
x_0 (ETH)	100	1 000
y_0 (USDC)	300 000	3 000 000
k	3×10^7	3×10^9
Slippage	9.09 %	0.99 %

Same spot price, 10× deeper pool → ∼9× less slippage.



Pool depth is the single most important factor for trade execution quality. Slippage $= \Delta x/(x_0 + \Delta x)$**: larger** x_0 **dominates the denominator.**

Why Path Independence Matters

Three practical implications of $\Delta y_{\text{total}} = f(\Delta x_{\text{total}})$ only:

1. No trade-splitting advantage

$$\sum_{i=1}^{n} \Delta y_i = y_0 - \frac{k}{x_0 + \sum_{i=1}^{n} \Delta x_i} = y_0 - \frac{k}{x_0 + \Delta x_{\text{total}}}$$

You cannot game the formula by splitting orders.

2. Simplifies analysis
Only the initial state (x_0, y_0) and total input Δx determine the output. Intermediate states are irrelevant.

3. A note on real AMMs
In production AMMs with trading costs, splitting **does** change the result — each sub-trade incurs costs separately. Path independence holds **only** in the idealized model where k is strictly constant during every trade.

Path independence is a direct consequence of the invariant $x \cdot y = k$ being state-dependent, not path-dependent.

Arbitrage Aligns the Pool to External Prices

Pool spot $= 3\,000$ USDC/ETH. External market (CEX) $= 3\,100$ USDC/ETH.

Arbitrageur buys cheap ETH from pool, sells on CEX. New equilibrium:

$$x_1 = \sqrt{\frac{k}{p_{\text{target}}}} = \sqrt{\frac{3 \times 10^9}{3\,100}} = 983.7388 \text{ ETH}$$

$$y_1 = \sqrt{k \cdot p_{\text{target}}} = \sqrt{3 \times 10^9 \times 3\,100} = 3\,049\,590.14 \text{ USDC}$$

Verify: $x_1 \cdot y_1 = 983.7388 \times 3\,049\,590.14 = 3\,000\,000\,000 = k$ ✓
$p_1 = y_1/x_1 = 3\,049\,590.14/983.7388 = 3\,100.00$ ✓

The square-root formulas come from solving $x_1 \cdot y_1 = k$ and $y_1/x_1 = p_{\text{target}}$ simultaneously.

Arbitrage Worked Example

Step 1: Buy ETH from pool

$$\text{ETH bought} = 1\,000 - 983.74 = 16.26 \text{ ETH}$$

$$\text{USDC in} = 3\,049\,590 - 3\,000\,000 = 49\,590 \text{ USDC}$$

$$p^{\text{AMM}}_{\text{exec}} = 49\,590.14/16.2612 = 3\,049.60$$

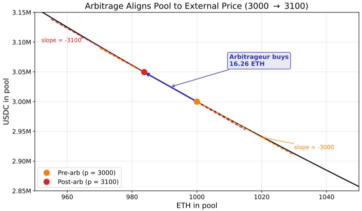
Step 2: Sell ETH on CEX at 3 100

$$\text{Revenue} = 16.2612 \times 3\,100 = 50\,409.72$$

$$\text{Gross profit} = 50\,409.72 - 49\,590.14$$

$$= \mathbf{819.58} \text{ USDC}$$

Buy: 3 049.60 Sell: 3 100.00 Spread: 50.40



Arbitrage keeps AMM prices aligned with external markets. The profit incentivizes bots to correct mispricings almost instantly.

LP Mint — Adding Liquidity

Deposit 50 ETH + 150 000 USDC into Pool A.

$$\text{Ratio check:} \quad \frac{150\,000}{50} = 3\,000 = p_0 \checkmark$$

$$\text{Share of existing pool:} \quad \frac{50}{1\,000} = 5\,\%$$

$$\text{New reserves:} \quad x_{\text{new}} = 1\,050 \text{ ETH}, \quad y_{\text{new}} = 3\,150\,000 \text{ USDC}$$

$$k_{\text{new}} = 1\,050 \times 3\,150\,000 = 3\,307\,500\,000 \quad (k \text{ changes — this is not a trade!})$$

$$p_{\text{new}} = \frac{3\,150\,000}{1\,050} = 3\,000 \text{ USDC/ETH (unchanged } \checkmark\text{)}$$

$$\text{LP tokens minted:} \quad 50 \quad \text{(total supply: } 1\,050\text{)}$$

$$\text{Depositor owns:} \quad \frac{50}{1\,050} = 4.7619\,\%$$

Adding liquidity changes k but preserves the spot price. The depositor's share ($4.76\,\%$) is less than $5\,\%$ because the denominator grew.

Summary of All Worked Examples

Example	Δx	**Pool**	Δy_{out} (USDC)	p_{exec}	Slippage
Slide 10: tiny	1 ETH	A	2 997.00	2 997.00	0.10 %
Slide 11: small	10 ETH	A	29 702.97	2 970.30	0.99 %
Slide 12: medium	100 ETH	A	272 727.27	2 727.27	9.09 %
Slide 13: large	500 ETH	A	1 000 000.00	2 000.00	33.33 %
Slide 19: split	3×1 ETH	A	8 973.08	2 991.03	0.30 %
Slide 19: single	3 ETH	A	8 973.08	2 991.03	0.30 %
Slide 20: shallow	10 ETH	B	27 272.73	2 727.27	9.09 %
Slide 22–23: arb	16.26 ETH bought		(49 590.14 in)	3 049.60	—
Slide 24: LP mint	50 ETH + 150k		$k_{\text{new}} = 3.308 \times 10^9$, share = 4.76 %		
App. A4: 250 ETH	250 ETH	A	600 000.00	2 400.00	20.00 %
App. A5: retail	0.1 ETH	A	299.97	2 999.70	0.01 %
App. A6: inverse	34.48 ETH	A	100 000.00	2 900.00	3.33 %

All examples use $p_0 = 3\,000$ USDC/ETH. Pool A: $k = 3 \times 10^9$. Pool B: $k = 3 \times 10^7$. Values rounded to 2 dp.

Edge Cases

Extreme sell: $\Delta x \to \infty$

$$p_{\text{exec}} = \frac{y_0}{x_0 + \Delta x} \to 0$$

$$\Delta y_{\text{out}} = \frac{y_0 \cdot \Delta x}{x_0 + \Delta x} \to y_0$$

You can **never** drain the USDC side completely. The asymptote is $\Delta y_{\text{out}} = y_0 = 3\,000\,000$.

Reverse direction: buying ETH
Deposit Δy USDC, receive ETH:

$$\boxed{\Delta x_{\text{out}} = \frac{x_0 \cdot \Delta y}{y_0 + \Delta y}}$$

Quick numerical: $\Delta y = 300\,000$ USDC:

$$\Delta x_{\text{out}} = \frac{1\,000 \times 300\,000}{3\,300\,000} = 90.9091 \text{ ETH}$$

$$p_{\text{exec}} = \frac{300\,000}{90.9091} = 3\,300 \text{ USDC/ETH}$$

The formula is symmetric: swap $x \leftrightarrow y$ and $\Delta x \leftrightarrow \Delta y$ and everything still works. Buying ETH means $p_{\text{exec}} > p_0$.

Key Takeaways

1. **One formula runs everything.** $x \cdot y = k$ gives us spot price (y/x), trade output $\left(\frac{y_0 \cdot \Delta x}{x_0 + \Delta x}\right)$, slippage, price impact, and LP share calculations.
2. **Slippage $\neq$ price impact.** Slippage $= \dfrac{\Delta x}{x_0 + \Delta x}$ measures the gap between quoted and executed price. Price impact $= 1 - \left(\frac{x_0}{x_0 + \Delta x}\right)^2$ measures the shift in marginal price. For small trades, PI $\approx 2 \times$ Slippage.
3. **Pool depth beats everything.** Same trade, $10\times$ bigger pool $\rightarrow \sim 10\times$ less slippage. Slippage $= \frac{\Delta x}{x_0 + \Delta x}$: larger x_0 dominates the denominator.
4. **Path independence.** In a CPMM without trading costs, splitting trades makes no difference to total output. Only the total trade size Δx matters.

These four facts are sufficient to analyze any constant-product AMM trade from first principles.

Appendix

Full Derivations and Extended Examples

Slides A1–A7

A1: Derivation — Slippage Closed Form

Goal: Show that Slippage $= \dfrac{\Delta x}{x_0 + \Delta x}$.

$$\begin{aligned}
\text{Slippage} &= 1 - \frac{p_{\text{exec}}}{p_0} = 1 - \frac{\Delta y_{\text{out}}/\Delta x}{y_0/x_0} = 1 - \frac{x_0}{\Delta x} \cdot \frac{\Delta y_{\text{out}}}{y_0} \\
&= 1 - \frac{x_0}{\Delta x} \cdot \frac{1}{y_0} \cdot \frac{y_0 \cdot \Delta x}{x_0 + \Delta x} \qquad \text{(substituting the trade formula)} \\
&= 1 - \frac{x_0}{\Delta x} \cdot \frac{\Delta x}{x_0 + \Delta x} = 1 - \frac{x_0}{x_0 + \Delta x} \\
&= \frac{x_0 + \Delta x - x_0}{x_0 + \Delta x} = \boxed{\frac{\Delta x}{x_0 + \Delta x}}
\end{aligned}$$

← Back to Slippage vs Price Impact. Note: slippage depends only on the ratio $\Delta x/x_0$, not on y_0 or k.

A2: Derivation — Price Impact Closed Form

Goal: Show that PI $= 1 - \left(\frac{x_0}{x_0 + \Delta x}\right)^2$.

$$p_1 = \frac{y_1}{x_1} = \frac{k/x_1}{x_1} = \frac{k}{x_1^2}$$

$$p_0 = \frac{y_0}{x_0} = \frac{k}{x_0^2} \qquad (\text{since } k = x_0 y_0 \text{ and } y_0 = k/x_0)$$

$$\frac{p_1}{p_0} = \frac{k/x_1^2}{k/x_0^2} = \frac{x_0^2}{x_1^2} = \left(\frac{x_0}{x_0 + \Delta x}\right)^2$$

$$\text{Price impact} = 1 - \frac{p_1}{p_0} = \boxed{1 - \left(\frac{x_0}{x_0 + \Delta x}\right)^2}$$

← **Back to Slippage vs Price Impact.** **Like slippage, price impact depends only on the ratio $\Delta x/x_0$.**

A3: Small-Trade Taylor Expansion

Let $\varepsilon = \Delta x / x_0 \ll 1$. Write $\dfrac{1}{1+\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \cdots$

Slippage:

$$s = \frac{\varepsilon}{1+\varepsilon} = \varepsilon\,(1 - \varepsilon + \varepsilon^2 - \cdots) \approx \varepsilon - \varepsilon^2$$

Price impact:

$$\begin{aligned} \mathrm{PI} = 1 - \frac{1}{(1+\varepsilon)^2} &= 1 - (1 - \varepsilon + \varepsilon^2 - \cdots)^2 \\ &= 1 - (1 - 2\varepsilon + 3\varepsilon^2 - \cdots) \approx 2\varepsilon - 3\varepsilon^2 \end{aligned}$$

Ratio:

$$\frac{\mathrm{PI}}{s} \approx \frac{2\varepsilon - 3\varepsilon^2}{\varepsilon - \varepsilon^2} = \frac{2 - 3\varepsilon}{1 - \varepsilon} \xrightarrow{\varepsilon \to 0} 2$$

Check: $\varepsilon = 0.1$ (100 ETH): $s \approx 0.1 - 0.01 = 0.09$, $\mathrm{PI} \approx 0.2 - 0.03 = 0.17$. Exact: $s = 9.09\,\%$, $\mathrm{PI} = 17.36\,\%$. ✓

← Back to main slides. **The $2\times$ rule is the leading-order relationship. Higher-order terms cause the ratio to fall below 2 for large trades.**

Proof of Path Independence — Step 1: The Key Identity

Claim: splitting a trade into pieces gives the same total USDC out.

Step 1: What does one sub-trade produce?
Before trade i: pool holds x_{i-1} ETH and $y_{i-1} = k/x_{i-1}$ USDC. Trader sells δ_i ETH. After: $x_i = x_{i-1} + \delta_i$. USDC out:

$$\Delta y_i = y_{i-1} - y_i = \frac{k}{x_{i-1}} - \frac{k}{x_i}$$

That is it. Each sub-trade's output is **the difference between two consecutive k/x values**.

Step 2: Write out all three trades from Slide 19 ($\delta_1 = \delta_2 = \delta_3 = 1$ ETH):

$$\begin{aligned}
\Delta y_1 &= \frac{k}{x_0} - \frac{k}{x_1} && = \frac{k}{1000} - \frac{k}{1001} = 2\,997.003 \\
\Delta y_2 &= \frac{k}{x_1} - \frac{k}{x_2} && = \frac{k}{1001} - \frac{k}{1002} = 2\,991.021 \\
\Delta y_3 &= \frac{k}{x_2} - \frac{k}{x_3} && = \frac{k}{1002} - \frac{k}{1003} = 2\,985.057
\end{aligned}$$

Look at the colors: k/x_1 appears once with $-$ and once with $+$, so it **cancels**. Same for k/x_2. Only the **first** and **last** terms survive.

← Back to Path Independence. The cancellation pattern is called a telescoping sum.

Proof of Path Independence — Step 2: Everything Cancels

Add up the three sub-trades (continuing from previous slide):

$$\Delta y_{\text{total}} = \left(\frac{k}{x_0} - \cancel{\frac{k}{x_1}}\right) + \left(\cancel{\frac{k}{x_1}} - \cancel{\frac{k}{x_2}}\right) + \left(\cancel{\frac{k}{x_2}} - \frac{k}{x_3}\right)$$

$$= \frac{k}{x_0} - \frac{k}{x_3} = \frac{k}{1\,000} - \frac{k}{1\,003} = 3\,000\,000 - 2\,991\,026.92 = \mathbf{8\,973.08}$$

This is exactly the same as one trade of $\Delta x = 3$ ETH: $\Delta y = k/x_0 - k/(x_0 + 3) = 8\,973.08$. ✓

General case (n sub-trades of any sizes):

$$\Delta y_{\text{total}} = \sum_{i=1}^{n} \left(\frac{k}{x_{i-1}} - \frac{k}{x_i} \right) = \frac{k}{x_0} - \frac{k}{x_n} = y_0 - \frac{k}{x_0 + \Delta x}$$

Every intermediate k/x_i cancels. The total depends only on the starting reserves and $\Delta x_{\text{total}} = \sum \delta_i$. ∎

When does this break? With fees: each sub-trade charges a fee, so k *grows* between trades. The k in Δy_2 is no longer the same k as in Δy_1, and the terms no longer cancel.

← **Back to Path Independence.** **The proof uses only $y = k/x$ and that k is constant. Any invariant with this property (not just $xy = k$) is path-independent.**

A4: Extended Example — Sell $\Delta x = 250$ ETH

$$x_1 = 1\,000 + 250 = 1\,250$$

$$y_1 = \frac{3\,000\,000\,000}{1\,250} = 2\,400\,000 \text{ USDC}$$

$$\Delta y_{\text{out}} = 3\,000\,000 - 2\,400\,000 = 600\,000 \text{ USDC}$$

$$p_{\text{exec}} = \frac{600\,000}{250} = 2\,400 \text{ USDC/ETH}$$

$$\text{Slippage} = \frac{250}{1\,250} = 20.00\,\%$$

$$\text{Price impact} = 1 - \left(\frac{1\,000}{1\,250}\right)^2 = 1 - 0.64 = 36.00\,\%$$

$$\text{Ratio} = 36.00/20.00 = 1.80$$

Verify: $1\,250 \times 2\,400\,000 = 3\,000\,000\,000 = k$ ✓

$25\,\%$ **of pool** → $20\,\%$ **slippage,** $36\,\%$ **price impact. New spot:** $p_1 = 3 \times 10^9/1\,250^2 = 1\,920$ **USDC/ETH.**

A5: Extended Example — Sell $\Delta x = 0.1$ ETH (Retail Size)

$$x_1 = 1\,000 + 0.1 = 1\,000.1$$

$$y_1 = \frac{3\,000\,000\,000}{1\,000.1} = 2\,999\,700.0300 \text{ USDC}$$

$$\Delta y_{\text{out}} = 3\,000\,000 - 2\,999\,700.0300 = 299.9700 \text{ USDC}$$

$$p_{\text{exec}} = \frac{299.9700}{0.1} = 2\,999.7000 \text{ USDC/ETH}$$

$$\text{Slippage} = \frac{0.1}{1\,000.1} = 0.0100\,\%$$

$$\text{Price impact} = 1 - \left(\frac{1\,000}{1\,000.1}\right)^2 = 0.0200\,\%$$

$$\text{Ratio} = 0.0200/0.0100 = 2.00$$

At retail sizes, the 2× rule holds almost exactly and slippage is negligible.

A typical retail swap in a $6M pool loses only 0.01 % **to slippage — less than** $0.03 **on a** $300 **trade.**

A6: Inverse Formula — How Much ETH for Exactly 100 000 USDC?

Problem: You want exactly $\Delta y_{\text{out}} = 100\,000$ USDC. How much ETH must you sell?

Invert the trade formula:

$$\Delta y_{\text{out}} = \frac{y_0 \cdot \Delta x}{x_0 + \Delta x}$$

$$\Delta y_{\text{out}}\,(x_0 + \Delta x) = y_0 \cdot \Delta x$$

$$\Delta y_{\text{out}} \cdot x_0 + \Delta y_{\text{out}} \cdot \Delta x = y_0 \cdot \Delta x$$

$$\Delta y_{\text{out}} \cdot x_0 = \Delta x\,(y_0 - \Delta y_{\text{out}})$$

$$\Delta x = \frac{\Delta y_{\text{out}} \cdot x_0}{y_0 - \Delta y_{\text{out}}} = \frac{100\,000 \times 1\,000}{3\,000\,000 - 100\,000} = \frac{100\,000\,000}{2\,900\,000} = 34.4828 \text{ ETH}$$

Verify: $\Delta y = \dfrac{3\,000\,000 \times 34.4828}{1\,000 + 34.4828} = \dfrac{103\,448\,275.86}{1\,034.4828} = 100\,000$ ✓

The inverse formula requires $\Delta y_{\text{out}} < y_0$. You cannot extract all the USDC — the hyperbola is asymptotic.

A7: Reading List

Primary sources:

1. H. Adams, N. Zinsmeister, D. Robinson. *Uniswap V2 Core.* Whitepaper, 2020.
   https://uniswap.org/whitepaper.pdf

2. G. Angeris, H.-T. Kao, R. Chiang, C. Noyes, T. Chitra. “An analysis of Uniswap markets.” *Cryptoeconomic Systems*, 2021.
   https://arxiv.org/abs/1911.03380

3. J. Xu, K. Paruch, S. Cousaert, Y. Feng. “SoK: Decentralized Exchanges (DEX) with Automated Market Maker (AMM) Protocols.” *ACM Computing Surveys*, 55(11), 2023.
   https://doi.org/10.1145/3570639

Notation note: Some papers write the invariant as $R_\alpha \cdot R_\beta = k$ or $x \cdot y = L^2$. The algebra is identical.

All three papers are freely accessible. The Angeris et al. paper introduced the formal framework that most subsequent AMM research builds on.